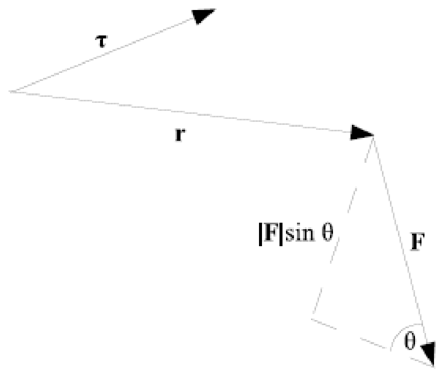
Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.3 **The cross product.**

If we tighten a bolt by applying a force to a wrench, we produce a turning effect called a *torque* $\vec{\tau}$ that acts along the axis of the bolt to move it forward.



The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is $|\vec{r}|$, the length of the position vector $\vec{r}$.
- The scalar component of the force $\vec{F}$ in the direction perpendicular to $\vec{r}$. This is the only component that can cause a rotation and it is

$$|\vec{F}|\sin\theta$$

where θ is an angle between the vectors $\vec{r}$ and $\vec{F}$.

We define the magnitude of the torque to be the product of these two factors:

$$|\tau| = |\vec{r}||\vec{F}|\sin\theta$$

If $\vec{n}$ is a unit vector that points in the direction in which a right-threaded bolt moves, we define the torque to be the vector

$$\vec{\tau} = (|\vec{r}||\vec{F}|\sin\theta)\vec{n}. \quad = \vec{r}\times\vec{F}$$

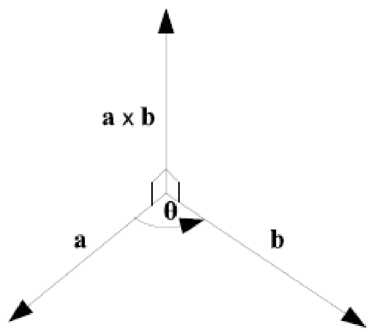
We denote this torque vector by $\vec{\tau} = \vec{r}\times\vec{F}$ and we call it the *cross product* or *vector product* of $\vec{r}$ and $\vec{F}$.

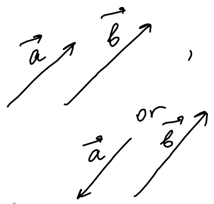
Definition. If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose direction is given by the right-hand rule: If the fingers of your hand curl through the angle θ from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{n}$.

1





, $\theta = 0 \Rightarrow \sin\theta = 0$

or

, $\theta = \pi \Rightarrow \sin\pi = 0$

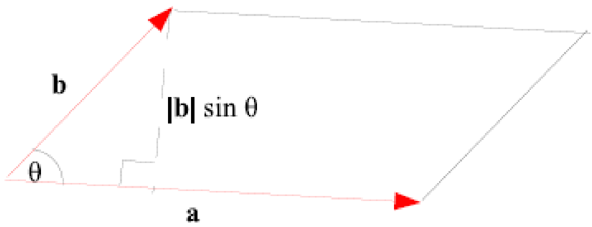
If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then we define $\vec{a} \times \vec{b}$ to be $\vec{0}$.

- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Properties of the cross product. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



Area = $|\vec{a}||\vec{b}| \sin\theta = |\vec{a}\times\vec{b}|$

The cross product in component form.

A determinant of order 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A determinant of order 3 can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}_2 \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix} \; \begin{matrix} a_3 \\ b_3 \\ c_3 \end{matrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

The cross product of a $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{\imath} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

Example 1. If $\vec{a} = < -2, 3, 4 >$ and $\vec{b} = < 3, 0, 1 >$, find $\vec{a} \times \vec{b}$.

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = \vec{\imath} \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix} = 3\vec{\imath} - \vec{\jmath}(-2-12) + \vec{k}(0-9)$$

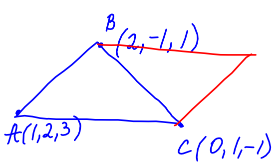
$$= 3\vec{\imath} + 14\vec{\jmath} - 9\vec{k} = \boxed{< 3, 14, -9 >}$$

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} \quad \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} \quad \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} \begin{matrix} \vec{\imath} & \vec{\jmath} \\ -2 & 3 \\ 3 & 0 \end{matrix} \; \begin{matrix} \vec{k} \\ 4 \\ 1 \end{matrix} = 3\vec{\imath} + 12\vec{\jmath} + 0 - 9\vec{k} - (-2)\vec{\jmath} - 0$$

$$= 3\vec{\imath} + 14\vec{\jmath} - 9\vec{k} = \boxed{< 3, 14, -9 >}$$

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.



area (parallelogram) $= |\vec{AB} \times \vec{AC}|$

area $(\triangle ABC) = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$\vec{AB} = < 2-1, -1-2, 1-3 > = < 1, -3, -2 >$

$\vec{AC} = < 0-1, 1-2, -1-3 > = < -1, -1, -4 >$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -3 & -2 \\ -1 & -1 & -4 \end{vmatrix} = \vec{\imath} \begin{vmatrix} -3 & -2 \\ -1 & -4 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 1 & -2 \\ -1 & -4 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}$$

$$= \vec{\imath}(12-2) - \vec{\jmath}(-4-2) + \vec{k}(-1-3)$$

$$= 10\vec{\imath} + 6\vec{\jmath} - 4\vec{k}$$

area $= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{10^2 + 6^2 + (-4)^2} = \frac{1}{2}\sqrt{100 + 36 + 16} = \boxed{\frac{1}{2}\sqrt{152}} = \frac{\sqrt{38 \cdot 4}}{2} = \boxed{\sqrt{38}}$

Example 3. Find two unit vectors orthogonal to both $\vec{\imath}+\vec{\jmath}$ and $\vec{\imath}-\vec{\jmath}+\vec{k}$.

$\vec{a}=\vec{\imath}+\vec{\jmath}=\langle 1,1,0\rangle$

$\vec{b}=\vec{\imath}-\vec{\jmath}+\vec{k}=\langle 1,-1,1\rangle$

$\vec{c}=\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{c}=\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 1 & 1 & 0\\ 1 & -1 & 1\end{vmatrix}=\vec{\imath}\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}-\vec{\jmath}\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}+\vec{k}\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}$$

$$=\vec{\imath}-\vec{\jmath}+\vec{k}(-1-1)=\vec{\imath}-\vec{\jmath}-2\vec{k}=\langle 1,-1,-2\rangle$$

$\vec{u}=\dfrac{\vec{c}}{|\vec{c}|}$ is the unit vector in the direction of $\vec{c}$.

$\vec{u}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{u}=\frac{\langle 1,-1,-2\rangle}{\sqrt{1^2+(-1)^2+(-2)^2}}=\frac{\langle 1,-1,-2\rangle}{\sqrt{6}}=\boxed{\left\langle \frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right\rangle}$$

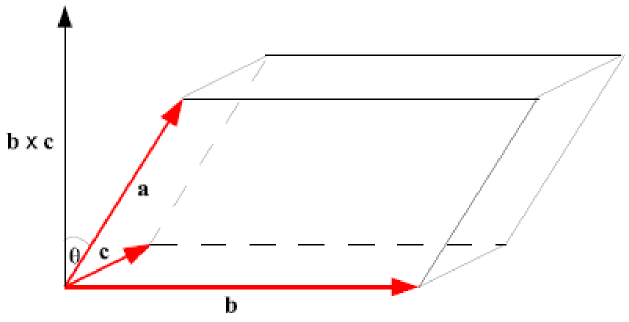
$$\vec{u}_2=-\vec{u}=\boxed{\left\langle -\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right\rangle}$$

Triple products

The product $\boxed{\vec{a}\cdot(\vec{b}\times\vec{c})}$ is called the scalar triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|.$$



$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given in component form:

$$\vec{a} =< a_1, a_2, a_3 >, \quad \vec{b} =< b_1, b_2, b_3 >, \quad \vec{c} =< c_1, c_2, c_3 > .$$

Then

$$\boxed{\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

4

Example 4. Find the volume of the parallelepiped determined by vectors $\vec{a} = 2\vec{\imath}+3\vec{\jmath}-2\vec{k}$, $\vec{b} = \vec{\imath}-\vec{\jmath}$, and $\vec{c} = 2\vec{\imath}+3\vec{k}$.

$\vec{a} = <2,3,-2>$, $\vec{b} = <1,-1,0>$, $\vec{c} = <2,0,3>$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \left|\begin{array}{ccc|ccc} 2 & 3 & -2 & 2 & 3 & -2 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & 0 & 3 & 2 & 0 & 3 \end{array}\right. = -6+0+0-4-9-0 = -19$$

$$\text{Volume} = |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-19| = \boxed{19}$$

Example 5. Use the scalar triple product to verify that the vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath}$, and $\vec{c} = 7\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$ are complanar; that is, they lie in the same plane.

if $\vec{a}, \vec{b}$ and $\vec{c}$ $\underbrace{\text{lie in the same plane}}_{\text{are complanar}}$ if and only if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

$\vec{a} = \langle 2, 3, 1\rangle$, $\vec{b} = \langle 1, -1, 0\rangle$, $\vec{c} = \langle 7, 3, 2\rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} \begin{matrix} 2 & 3 \\ 1 & -1 \\ 7 & 3 \end{matrix} = -4 + 0 + 3 - (-7) - 6 - 0$$

$$= 0$$

$\boxed{\text{YES}}$

The product $\vec{a} \times (\vec{b} \times \vec{c})$ is called the vector triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.}$$