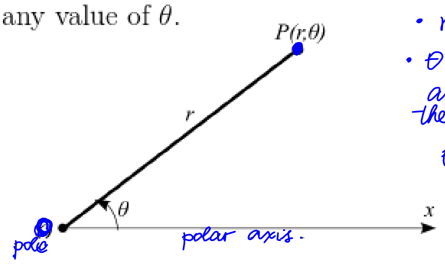
Section 13.4 Polar coordinates

We choose a point in the plane that is called the **pole** (or origin) and labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drown horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any point in the plane, let r be the distance from O to P and let θ be the angle (in radians) between the polar axis and the line OP. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P.

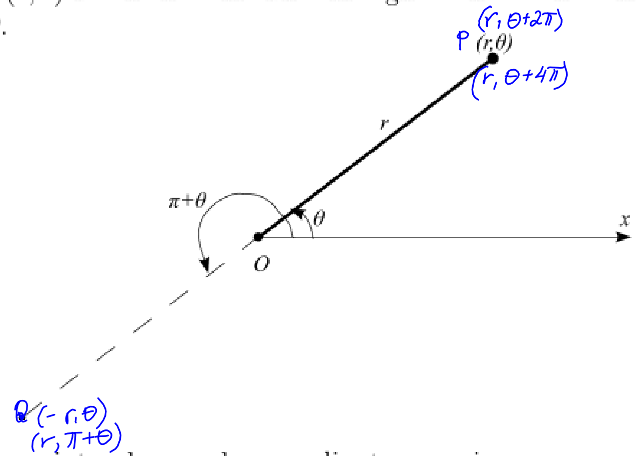
We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = 0$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.



- r is the distance from P to O
- θ is the angle between PO and the positive direction of the polar axis.

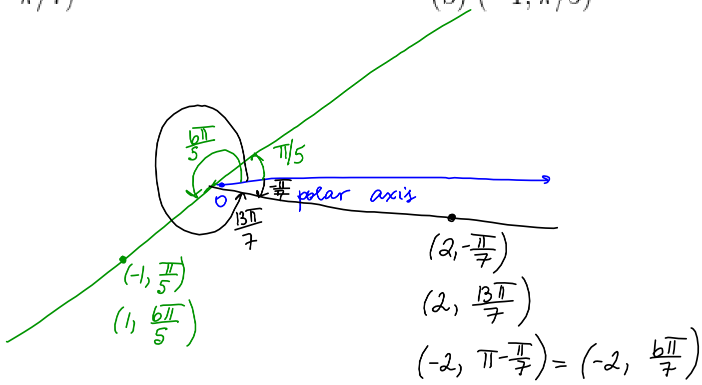
θ is in radians.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.



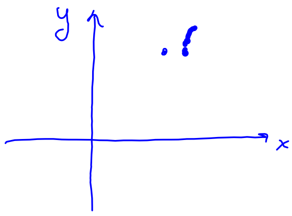
Example 1. Plot the points whose polar coordinates are given:

(a) $(2, -\pi/7)$ (b) $(-1, \pi/5)$



In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. Since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2\pi n) \quad \text{and} \quad (-r, \theta + (2n+1)\pi),$$



where n is any integer.

The connection between polar and Cartesian coordinates is

$$x = r\cos\theta \quad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}.$$

Equation for θ do not uniquely determine it when x and y are given. Therefore, in converting from Cartesian to polar coordinates, it is not good enough just to find r and θ that satisfy equations. We must choose θ so that the point (r, θ) lies in correct quadrant.

Example 2. Convert the point $(2, 2\pi/3)$ from polar to Cartesian coordinates.

Example 3. Represent the point with Cartesian coordinates $(-1, -\sqrt{3})$ in terms of polar coordinates.

The graph of a polar equation $r = f(\theta)$, or more generally, $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.
Note that:

1. If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
2. If the equation is unchanged when r is replaced by $-r$, the curve is symmetric about the pole.

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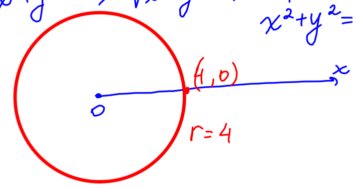
3. If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

Example 4. Graph the curves in polar coordinates.

1. $r = 4$

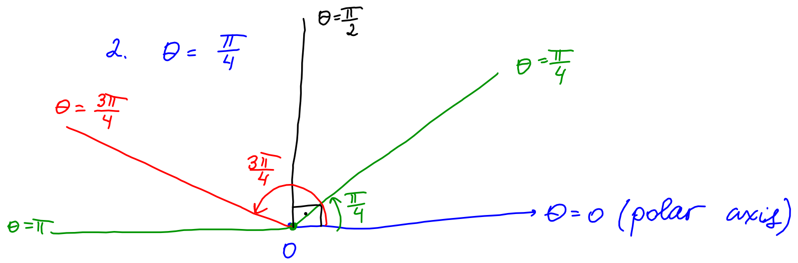
$r^2 = x^2+y^2 \Rightarrow \sqrt{x^2+y^2} = r = 4$

$x^2+y^2=16$ — circle centered @ $(0,0)$ of radius 4



$r=a$ $(a>0)$ is a circle of radius a centered at the pole.

2. $\theta = \frac{\pi}{4}$



$\theta = b$ — line through the pole that makes angle b with the positive direction of the polar axis

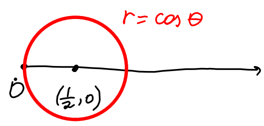
3. $r = \cos\theta$

Convert the equation into cartesian coordinates.

$r(r) = r(\cos\theta)$

$r^2 = r\cos\theta$

$r^2 = x^2+y^2, \quad x = r\cos\theta$



$x^2+y^2 = x$

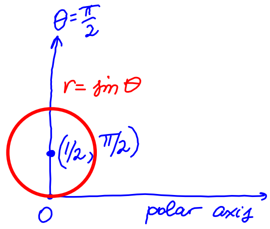
$x^2 - \frac{1}{2}\cdot 2x$

$x^2 - x + y^2 = 0$

$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + y^2 = 0 \Longrightarrow \left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$

circle of radius $\frac{1}{2}$ centered @ $\left(\frac{1}{2}, 0\right)$

4. $r = \sin\theta$ convert into cartesian coordinates.



$$\underbrace{r^2}_{x^2+y^2} = \underbrace{r\sin\theta}_{y}$$

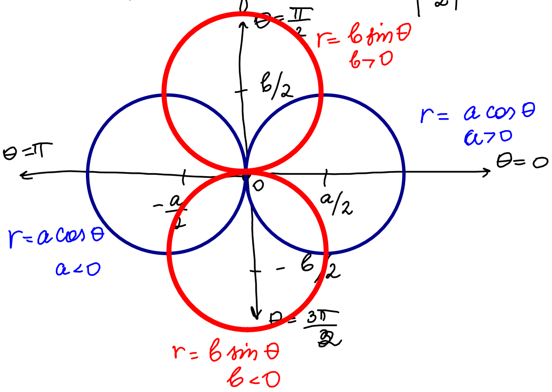
$$x^2+y^2 = y$$

$$x^2+y^2-y=0$$

$$x^2+\left(y^2-y+\tfrac{1}{4}\right)-\tfrac{1}{4}=0 \Rightarrow x^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$

circle centered at $(0,\frac{1}{2})$ of radius $\frac{1}{2}$

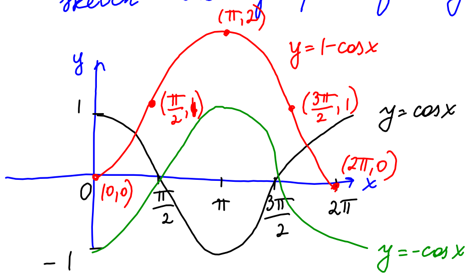
- $r = a\cos\theta \quad (a>0)$
  circle of radius $\frac{a}{2}$ centered at $(\frac{a}{2}, 0)$
- $r = a\cos\theta \quad (a<0)$
  circle of radius $\left|\frac{a}{2}\right|$ centered at $(\frac{a}{2}, \pi)$ or $(-\frac{a}{2}, 0)$
- $r = b\sin\theta \quad (b>0)$
  circle of radius $\frac{b}{2}$ centered at $(\frac{b}{2}, \frac{\pi}{2})$
- $r = b\sin\theta \, (b<0)$
  circle of radius $\left|\frac{b}{2}\right|$ centered at $(\frac{b}{2}, \frac{3\pi}{2})$ or $(-\frac{b}{2}, \frac{\pi}{2})$



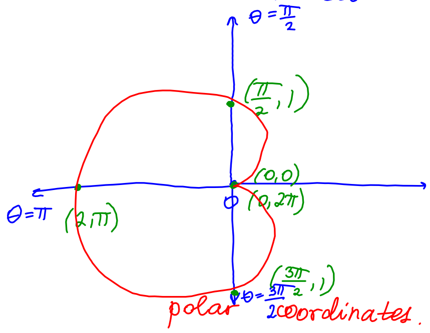
~~**Example 4.** Sketch the curve of each polar equation~~

~~(a)~~ $r = 1 - \cos\theta$

Sketch the graph of $y = 1-\cos x$ in the cartesian coordinates.



cartesian coordinates

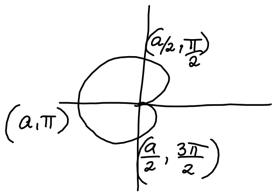


polar coordinates.

mark the points $(0,0)$, $(1,\frac{\pi}{2})$, $(2,\pi)$, $(1,\frac{3\pi}{2})$, $(0,2\pi)$

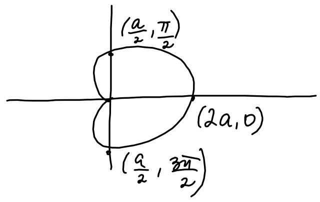
Cardioid $r = 1-\cos\theta$

- $r = a - a\cos\theta$

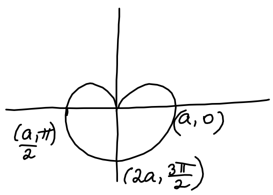


~~(b) $r = a\sin 2\theta$~~

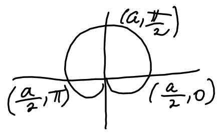
- $r = a + a\cos\theta$



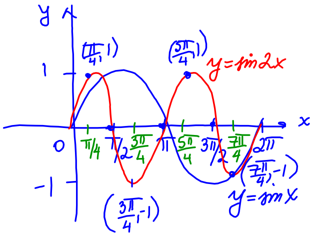
- $r = a - a\sin\theta$



- $r = a + a\sin\theta$

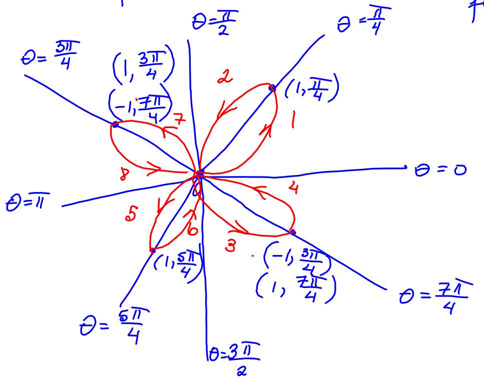


(6) $r = \sin 2\theta$.

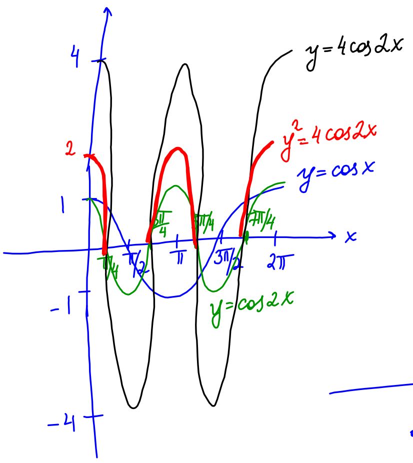


polar coordinates

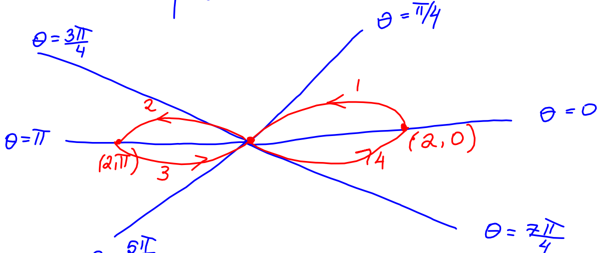
four-petal rose.



(7) $r^2 = 4\cos 2\theta$

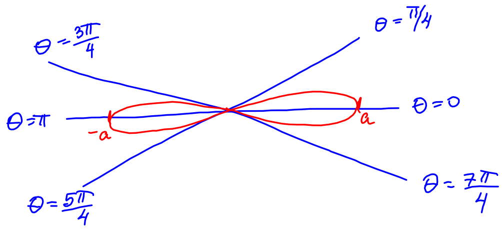


polar coordinates

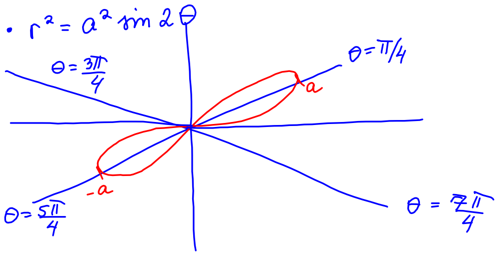


lemniscate

• $r^2 = a^2 \cos 2\theta$



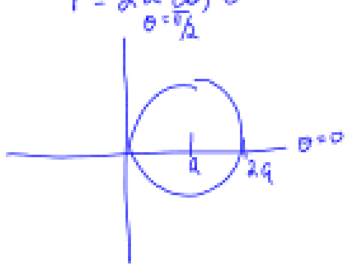
• $r^2 = a^2 \sin 2\theta$



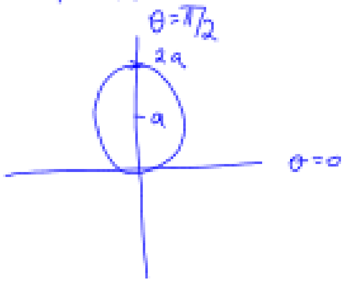
$a > 0$

Circles:

$r = 2a\cos\theta$

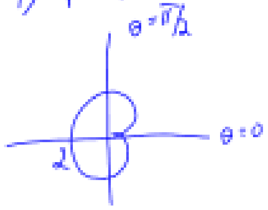


$r = 2a\sin\theta$

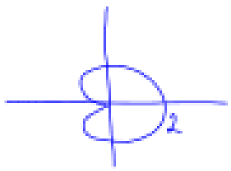


Cardioids:

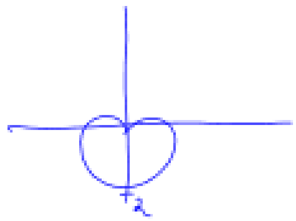
1) $r = 1-\cos\theta$



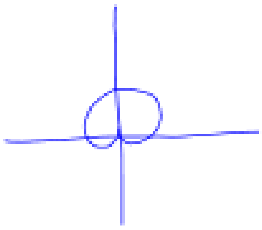
2) $r = 1+\cos\theta$



3) $r = 1-\sin\theta$

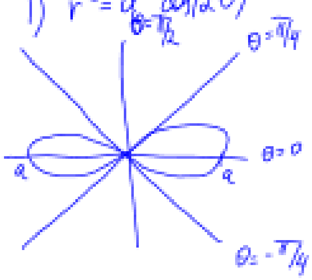


4) $r = 1+\sin\theta$



Lemniscate:

1) $r^2 = a^2\cos(2\theta)$



2) $r^2 = a^2\sin(2\theta)$