Table of indefinite integrals

1. $\int a dx = ax + C,\ a$ is a constant,
2. $\int x dx = \frac{x^2}{2} + C,$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1,$
4. $\int \frac{1}{x} dx = \ln|x| + C,$
5. $\int \mathrm{e}^x dx = \mathrm{e}^x + C,$
6. $\int \mathrm{a}^x dx = \frac{\mathrm{a}^x}{\ln \mathrm{a}} + C,$
7. $\int \sin x dx = -\cos x + C,$
8. $\int \cos x dx = \sin x + C,$
9. $\int \tan x dx = -\ln|\cos x| + C,$
10. $\int \cot x dx = \ln|\sin x| + C,$
11. $\int \sec^2 x dx = \tan x + C,$
12. $\int \csc^2 x dx = -\cot x + C,$
13. $\int \sec x \tan x dx = \sec x + C,$
14. $\int \csc x \cot x = -\csc x + C,$
15. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C,$
16. $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Definition of a definite integral

If f is a function defined on a closed interval $[a,b]$, let P be a partition of $[a,b]$ with partition points x_0, x_1,...,x_n, where

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

Choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a,b]$.

In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the limits of integration; a is the **lower limit** and b is the **upper limit**.

The procedure of calculating an integral is called **integration**.

Properties of the definite integral

1. $\int_a^b c\,dx = c(b-a)$, where c is a constant.
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant.
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

Section 6.4 The fundamental theorem of calculus.

Suppose f is continuous on $[a,b]$.

1. If $g(x)=\int_a^x f(t)dt$, then $g'(x)=f(x)$.
2. $\boxed{\int_a^b f(x)dx = F(b)-F(a) = F(x)\Big|_a^b}$ where F is an antiderivative of f.

Example 1. Evaluate the integral.

1. $\int_2^6 \frac{1+\sqrt{y}}{y^2}dy = \int_2^6 (1+\sqrt{y})y^{-2}\,dy = \int_2^6 (y^{-2}+\underbrace{\sqrt{y}\,y^{-2}}_{\underbrace{y^{1/2}y^{-2}}_{\underbrace{y^{1/2-2}}_{y^{-3/2}}}})dy$

$\boxed{\int y^n dy = \frac{y^{n+1}}{n+1}+C \quad n\neq -1}$

$= \int_2^6 (y^{-2}+y^{-3/2})\,dy = \left[\frac{y^{-1}}{-1}+\frac{y^{-3/2+1}}{-3/2+1}\right]_2^6 = \left[-y^{-1}+\frac{y^{-1/2}}{-1/2}\right]_2^6$

$= \left[-\frac{1}{y}-2\frac{1}{y^{1/2}}\right]_2^6 = \boxed{-\frac{1}{6}-2\frac{1}{6^{1/2}}-\left(-\frac{1}{2}-2\frac{1}{2^{1/2}}\right)}$

2. $\int_0^2 f(x)dx$, where $f(x)=\begin{cases} x^4 & 0\le x<1 \\ x^5 & 1\le x\le 2\end{cases}$

$= \int_0^1 \overbrace{f(x)}^{x^4}dx + \int_1^2 \overbrace{f(x)}^{x^5}dx = \int_0^1 x^4dx + \int_1^2 x^5dx$

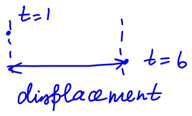
$= \frac{x^{4+1}}{4+1}\Big]_0^1 + \frac{x^{5+1}}{5+1}\Big]_1^2$

$= \frac{x^5}{5}\Big]_0^1 + \frac{x^6}{6}\Big]_1^2 = \frac{1}{5}-0+\frac{2^6}{6}-\frac{1}{6}$

$= \boxed{\frac{1}{5}+\frac{64}{6}-\frac{1}{6}}$

Example 2. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$.

1. Find the displacement of the particle during the time period $1 \leq t \leq 6$.



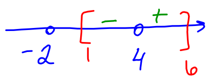
$$\text{displacement} = \left|\int_1^6 v(t)\,dt\right| = \left|\int_1^6 (t^2-2t-8)\,dt\right| = \left|\left[\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} - 8t\right]_1^6\right|$$

$$= \left|\frac{6^3}{3} - 6^2 - 8(6) - \left(\frac{1}{3} - 1 - 8\right)\right| = \left|72 - 36 - 48 - \frac{1}{3} + 9\right|$$

2. Find the distance traveled during this time period.

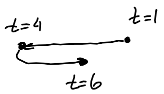
$v(t) = t^2 - 2t - 8 > 0$

$(t-4)(t+2) > 0$



$v(5) = (5-4)(5+2) = 7 > 0$

$v(2) = (2-4)(2+2) = -8 < 0$



$v(t) < 0$ on $[1, 4)$

$v(t) > 0$ on $(4, 6]$

$$\text{distance} = -\int_1^4 v(t)\,dt + \int_4^6 v(t)\,dt = -\int_1^4 (t^2-2t-8)\,dt + \int_4^6 (t^2-2t-8)\,dt$$