Section 6.5 **The Substitution Rule**

The substitution rule for indefinite integrals. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du}$$

$u = g(x)$
$g'(x)dx = du$

Example 1. Evaluate each integral:

1. $\int \sin 5x\, dx$ $\quad \frac{du}{5}$

$u = 5x$
differentiate: $(u)'du = (5x)'dx$
$du = 5dx$
$dx = \frac{du}{5}$

$$= \int \sin u \frac{du}{5} = \frac{1}{5}\int \sin u\, du = \frac{1}{5}(-\cos u) + C = \boxed{-\frac{1}{5}\cos(5x) + C}$$

2. $\frac{1}{3}\int \frac{3\,dx}{\sqrt{3x+1}}$

$u = 3x+1$
$du = (3x+1)'dx$
$du = 3dx$

$$= \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{1}{3}\frac{u^{1/2}}{1/2} + C = \frac{2}{3}u^{1/2} + C = \boxed{\frac{2}{3}(3x+1)^{1/2} + C}$$

3. $\frac{1}{3}\int 3x^2 e^{x^3} dx$

$u = x^3$
$du = (x^3)'dx$
$du = 3x^2 dx$

$$= \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \boxed{\frac{1}{3}e^{x^3} + C}$$

4. $\int \frac{2x^2 + 4x}{x^3 + 3x^2 - 4}dx$

$(x^3 + 3x^2 - 4)' = 3x^2 + 6x$
$= 3(x^2 + 2x)$

$u = x^3 + 3x^2 - 4$
$du = 3(x^2 + 2x)dx$

$$= \frac{1}{3}\int \frac{2(x^2+2x)(3)dx}{x^3+3x^2-4} = \frac{1}{3}\int \frac{2du}{u} = \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C = \boxed{\frac{2}{3}\ln|x^3+3x^2-4| + C}$$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

5. $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{\overbrace{-2x\,dx}^{du}}{\sqrt{\underbrace{1-x^2}_{=u}}} + \int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

$\left| \begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array} \right|$

$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$

$= -\frac{1}{2}\int u^{-1/2}du$

$= -\frac{1}{2}\frac{u^{-1/2+1}}{-1/2+1} + C$

$= -\cancel{\frac{1}{2}}\frac{u^{1/2}}{\cancel{1/2}} + C$

$= -u^{1/2} + C = -(1-x^2)^{1/2} + C$

$= \int \overbrace{\arcsin x}^{u} \overbrace{\frac{1}{\sqrt{1-x^2}}dx}^{du}$ $\left| \begin{array}{l} u = \arcsin x \\ du = \frac{1}{\sqrt{1-x^2}}dx \end{array} \right|$

$= \int u\,du = \frac{u^2}{2} + C = \frac{(\arcsin x)^2}{2} + C$

$$\boxed{-(1-x^2)^{1/2} + \frac{(\arcsin x)^2}{2} + C}$$

6. $\int x^5\sqrt{1+x^3}\,dx$ $\left| \begin{array}{l} u = 1+x^3 \longrightarrow x^3 = u-1 \\ du = 3x^2dx \\ x^5 = (x^3)(x^2) \end{array} \right|$ $= \frac{1}{3}\int \underbrace{(x^3)}_{u-1} \overbrace{(3x^2)\sqrt{\underbrace{1+x^3}_{u}}\,dx}^{=du}$

$= \frac{1}{3}\int (u-1)\sqrt{u}\,du = \frac{1}{3}\int (u^{3/2} - u^{1/2})\,du = \frac{1}{3}\left[\frac{u^{3/2+1}}{3/2+1} - \frac{u^{1/2+1}}{1/2+1}\right] + C$

$= \frac{1}{3}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C = \boxed{\frac{1}{3}\left[\frac{2}{5}(1+x^3)^{5/2} - \frac{2}{3}(1+x^3)^{3/2}\right] + C}$

The substitution rule for definite integrals. If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

$u = g(x)$
$du = g'(x)\,dx$
$a \longmapsto g(a)$
$b \longmapsto g(b)$

Example 2. Evaluate the integral:

1. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} =$

$u = \ln x$, $du = \frac{1}{x}dx$, $e \to \ln(e) = 1$, $e^4 \to \ln(e^4) = 4$

$$= \int_1^4 \frac{du}{\sqrt{u}} = \int_1^4 u^{-1/2}\,du = \frac{u^{-1/2+1}}{-1/2+1}\Big]_1^4$$

$$= \frac{u^{1/2}}{1/2}\Big]_1^4 = 2u^{1/2}\Big]_1^4$$

$$= 2\left(4^{1/2} - 1^{1/2}\right) = 2(2-1) = \boxed{2}$$

2. $\displaystyle\frac{1}{2}\int_0^1 \frac{2x\,dx}{\sqrt{1+x^4}}$

$u = x^2$, $du = 2x\,dx$, $0 \to 0^2 = 0$, $1 \to 1^2 = 1$

$$= \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1+u^2}} = \frac{1}{2}\ln\left|u + \sqrt{u^2+1}\right|\Big]_0^1$$

$$= \frac{1}{2}\left(\ln\left|1+\sqrt{1+1}\right| - \ln\left|0+\sqrt{0+1}\right|\right)$$

$$= \boxed{\frac{1}{2}\ln\left|1+\sqrt{2}\right|}$$

$$\boxed{\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right| + C}$$

3. $-\frac{1}{2}\int_{-3/2}^{0} \frac{-2x\,dx}{\sqrt{1-2x}} =$ $\left|\begin{array}{l} u = 1-2x \longrightarrow x = \frac{1-u}{2} \\ du = -2\,dx \\ 0 \to 1-2(0) = 1 \\ -3/2 \longrightarrow 1-2\left(-\frac{3}{2}\right) = 4 \end{array}\right|$ $= -\frac{1}{2}\int_4^1 \frac{\frac{1-u}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int_1^4 \frac{(1-u)\,du}{2\sqrt{u}}$

($0 \to 1$, $-3/2 \to 4$, $x \to \frac{1-u}{2}$, $-2\,dx \to du$, $1-2x \to u$)

$$= \frac{1}{4}\int_1^4 (1-u)\,u^{-1/2}\,du = \frac{1}{4}\int_1^4 \left(u^{-1/2} - u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{u^{-1/2+1}}{-1/2+1} - \frac{u^{1/2+1}}{1/2+1}\right)\Big|_1^4 = \frac{1}{4}\left(\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right)\Big|_1^4$$

$$= \frac{1}{4}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^4 = \frac{2}{4}\left(4^{1/2} - \frac{1}{3}4^{3/2} - 1^{1/2} + \frac{1}{3}1^{3/2}\right)$$

$$= \boxed{\frac{1}{2}\left(2 - \frac{8}{3} - 1 + \frac{1}{3}\right)}$$

Integrals of symmetric functions. Suppose f is continuous on $[-a, a]$.

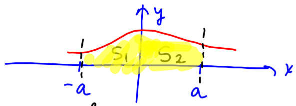
- If f is even, then $\boxed{\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx}$

  $f(-x) = f(x)$

- If f is odd, then $\boxed{\int_{-a}^{a} f(x)dx = 0}$

  $f(-x) = -f(x)$

  $x, x^3, x^5, \sin x, \tan x$

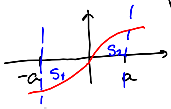


$f(x) > 0$ on $[-a, a]$

Area(S_1) = Area(S_2)

$\int_{-a}^{a} f(x)dx =$ Area(S_1) + Area(S_2)

$= 2$ Area(S_2) $= 2\int_0^a f(x)dx$



$\int_{-a}^{a} f(x)dx =$ area(S_2) − area(S_1) $= 0$

area(S_1) = area(S_2)

(odd)(odd) = even

(even)(odd) = odd

Example 3. Evaluate the integral $\boxed{\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}dx. = 0}$

$\sin x$ − odd

x^2 − even

$1+x^6$ − even

$\frac{x^2 \sin x}{1+x^6}$ − odd