Section 7.2 Volume

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1 , called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2 . If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as V = Ah.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is A(x), where $a \le x \le b$.

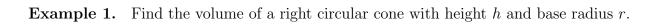
Let's consider a partition P of [a, b] by points x_i such that $a = x_0 < x_1 < ... < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

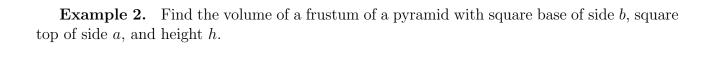
The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the *i*th slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $V \approx \sum_{i=1}^n A(x_i^*)\Delta x_i$. This approximation appears to become better and better as $||P|| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b . If the cross-sectional area of S in the plane P_x is A(x), where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*) \Delta x_i = \int_{a}^{b} A(x) dx$$

IMPORTANT. A(x) is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.





Volume by disks. Let S be the solid obtained by revolving the plane region \mathcal{R} bounded by y = f(x), y = 0, x = a, and x = b about the x-axis.

A cross-section through x perpendicular to the x-axis is a circular disc with radius |y| = |f(x)|, the cross-sectional area is $A(x) = \pi y^2 = \pi [f(x)]^2$, thus, we have the following formula for a volume of revolution:

$$V_X = \pi \int_a^b [f(x)]^2 dx$$

The region bounded by the curves x = g(y), x = 0, y = c, and y = d is rotated about the y-axis.

Then the corresponding volume of revolution is

$$V_Y = \pi \int_{c}^{d} [g(y)]^2 dy$$

Volume by washers. Let S be the solid generated when the region bounded by the curves $y=f(x),\ y=g(x),\ x=a,$ and x=b (where $f(x)\geq g(x)$ for all x in [a,b]) is rotated about the x-axis.

Then the volume of S is

$$V_X = \pi \int_{a}^{b} \{ [f(x)]^2 - [g(x)]^2 \} dx$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, x = 2y about the x-axis.

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, y = 0, x = 5 about the y-axis.

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, y = 1 about the line y = 2.