Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2. If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as $V = Ah$.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is $A(x)$, where $a \le x \le b$.

Let's consider a partition P of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the ith slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

IMPORTANT. $A(x)$ is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.

Example 1. Find the volume of a right circular cone with height h and base radius r.

Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.

Volume by disks. Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis.

A cross-section through x perpendicular to the x-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$V_X = \pi \int_a^b [f(x)]^2 dx$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis.

Then the corresponding volume of revolution is

$$V_Y = \pi \int_c^d [g(y)]^2 dy$$

Volume by washers. Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis.

Then the volume of S is

$$V_X = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the x-axis.

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the y-axis.

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.