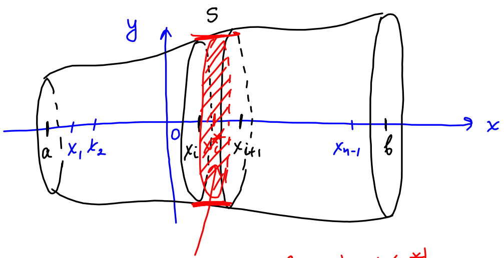
Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2. If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as $V = Ah$.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is $A(x)$, where $a \le x \le b$.



area of the slice is $A(x_i^*)$

Partition $[a,b]$ into n subintervals.
Partition points
$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

So planes P_{x_i} and $P_{x_{i+1}}$ perpendicular to the x-axis and pass thougth $x = x_i$ and $x = x_{i+1}$ respectively.

Take a point $x_i < x_i^* < x_{i+1}$ and approximate the "slice" of the solid by the cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_{i+1} - x_i$

Volume of the part of the solid $x_i \le x \le x_{i+1}$

$\boxed{V_i \approx A(x_i^*)\Delta x_i}$

Let's consider a partition P of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \dots < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

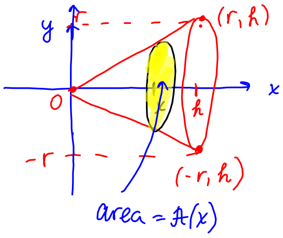
The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the ith slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \boxed{\int_a^b A(x)\,dx = V}$$

IMPORTANT. $\boxed{A(x)}$ is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.

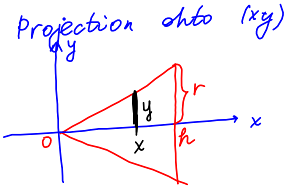
Example 1. Find the volume of a right circular cone with height h and base radius r.



Take an arbitrary $0<x<h$
Do a cross-section by the plane perpendicular to the x-axis through x.

$$V=\int_0^h A(x)\,dx$$

Projection onto (xy)-plane



similar triangles:

$$\frac{y}{x}=\frac{r}{h}$$

$$y=\frac{r}{h}x$$

area $= A(x) = \pi y^2$

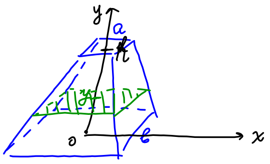
$$=\pi\left(\frac{r}{h}x\right)^2$$

$$A(x)=\pi\frac{r^2}{h^2}x^2$$

$$V=\int_0^h \pi\frac{r^2}{h^2}x^2\,dx=\pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$=\pi\frac{r^2}{h^2}\frac{h^3}{3}=\boxed{\frac{\pi h r^2}{3}}$$

Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.



integrate for y! $\quad 0 \le y \le h$

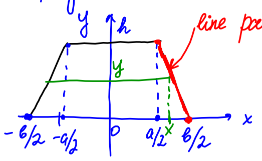
Slice the pyramid with a plane perpendicular to the y-axis through y. Find the area of the cross-section $A(y)$.

cross-section:

square of side x.

Area $=(2x)^2$

Projection onto the (xy)-plane



line passing through $(b/2,0)$ and $(\frac{a}{2},h)$

Express x in terms of y.

Eqn of the line through $(\frac{b}{2},0)$ and $(\frac{a}{2},h)$

slope $= \frac{h}{\frac{a}{2}-\frac{b}{2}} = \frac{2h}{a-b}$

$y-0 = \frac{2h}{a-b}(x-b/2)$ - solve for x.

$\left(y = \frac{2h}{a-b}(x-b/2)\right)\frac{a-b}{2h}$

$y\frac{a-b}{2h} = x-\frac{b}{2} \Rightarrow \boxed{x = \frac{a-b}{2h}y+\frac{b}{2}}$

$2x = \frac{a-b}{h}y+b$

$$\text{area} = A(y) = (2x)^2 = \left(\frac{a-b}{h}y+b\right)^2$$

$$V = \int_0^h A(y)dy = \int_0^h\left(\frac{a-b}{h}y+b\right)^2 dy = \int_0^h\left[\frac{(a-b)^2}{h^2}\boxed{y^2}+2\frac{a-b}{h}\boxed{y}b+b^2\right]dy$$

$$=\left[\frac{(a-b)^2}{h^2}\frac{y^3}{3}+\cancel{2}\frac{a-b}{h}\frac{y^2}{\cancel{2}}b+b^2y\right]_0^h$$

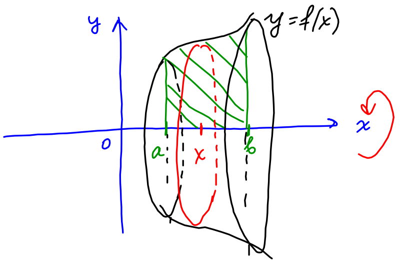
$$=\frac{(a-b)^2}{\cancel{h^2}}\cdot\frac{h^{\cancel{3}}}{3}+\frac{a-b}{\cancel{h}}h^{\cancel{2}}b+b^2h = \frac{(a-b)^2}{3}h+\overbrace{(a-b)hb}^{hab-hb^2}+b^2h$$

$$=\frac{(a-b)^2}{3}h+hab-\cancel{hb^2}+\cancel{b^2h} = \frac{a^2-2ab+b^2}{3}h+hab$$

$$=\frac{(a^2-2ab+b^2)h+3hab}{3} = \frac{h}{3}(a^2-2ab+b^2+3ab)$$

$$=\boxed{\frac{h}{3}(a^2+ab+b^2)}$$

Volume by disks. Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis.



slice the solid by a plane perpendicular to the x-axis through an arbitrary $a \le x \le b$

Cross-section is the circle of radius $f(x)$

$f(x)$

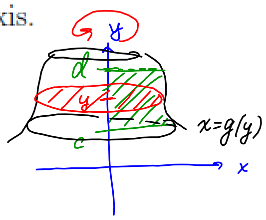
Area $= \pi (f(x))^2 = A(x)$

$V = \int_a^b A(x)\,dx$

A cross-section through x perpendicular to the x-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$\boxed{V_X = \pi \int_a^b [f(x)]^2 dx}$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis.

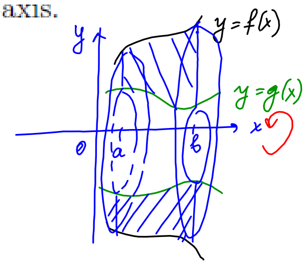


Cross-sectional by a plane perpendicular to the y-axis through an arbitrary $c \le y \le d$ is a circle of radius $g(y)$

Then the corresponding volume of revolution is

$$\boxed{V_Y = \pi \int_c^d [g(y)]^2 dy}$$

Volume by washers. Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis.



Cross-sectional is a washer
with inner radius $g(x)$
outer radius $f(x)$

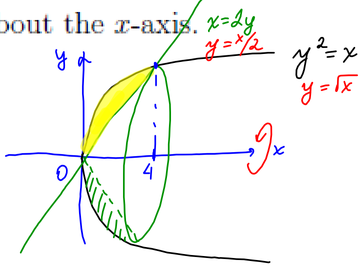
Area $= \pi(OR)^2 - \pi(IR)^2 = \pi\left((OR)^2 - (IR)^2\right)$
$OR =$ outer radius $=$ [top function]
$IR =$ inner radius $=$ [bottom function]

Then the volume of S is

$$\boxed{V_X = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx}$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the x-axis.



cross-sectional: washer

$[IR] = \frac{x}{2}$

$[OR] = \sqrt{x}$

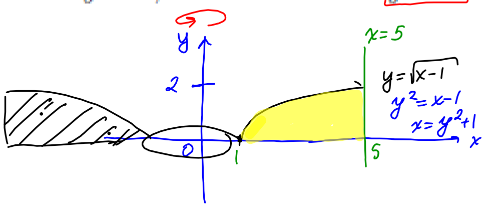
Points of intersection:
$y^2 = 2y \Rightarrow y(y-2) = 0$
$y = 0$ or $y = 2$
$x = y^2 = 0$ or $x = y^2 = 4$
$0 \leq x \leq 4$

$V_x = \pi \int_0^4 \left([OR]^2 - [IR]^2 \right) dx = \pi \int_0^4 \left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 \right) dx$

$= \pi \int_0^4 \left(x - \frac{x^2}{4} \right) dx = \pi \left(\frac{x^2}{2} - \frac{x^3}{12} \right)_0^4$

$= \boxed{\pi\left(\frac{16}{2} - \frac{64}{12}\right)}$

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the y-axis.



Integrate for y. $1 \le x \le 5$, $0 \le y \le 2$, $y = \sqrt{x-1}$

cross-sectional washer

$[IR] = y^2+1$

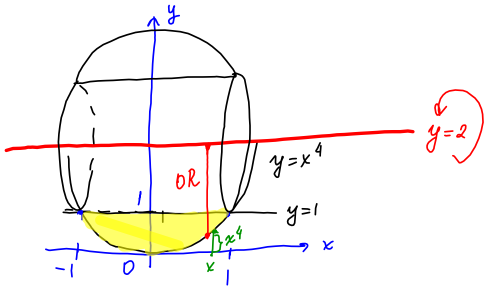
$[OR] = 5$

$$V_y = \pi \int_0^2 \left[5^2 - (y^2+1)^2 \right] dy = \pi \int_0^2 \left[25 - (y^4 + 2y^2 + 1) \right] dy$$

$$= \pi \int_0^2 (24 - y^4 - 2y^2)\, dy = \pi \left(24y - \frac{y^5}{5} - \frac{2y^3}{3} \right)\Big|_0^2$$

$$= \boxed{\pi \left(48 - \frac{32}{5} - \frac{16}{3} \right)}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.



$y=2$ is parallel to the x-axis

Integrate for x., $-1 \le x \le 1$

$$V = \pi \int_{-1}^{1} \left([OR]^2 - [IR]^2 \right) dx$$

$[OR] = 2 - x^4$

$[IR] = 1$

$$2\pi \int_0^1 \left((2-x^4)^2 - 1 \right) dx = \pi \int_{-1}^{1} \left((2-x^4)^2 - 1 \right) dx = \pi \int_{-1}^{1} \left(4 - 4x^4 + x^8 - 1 \right) dx$$

$$= \ldots$$