Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).

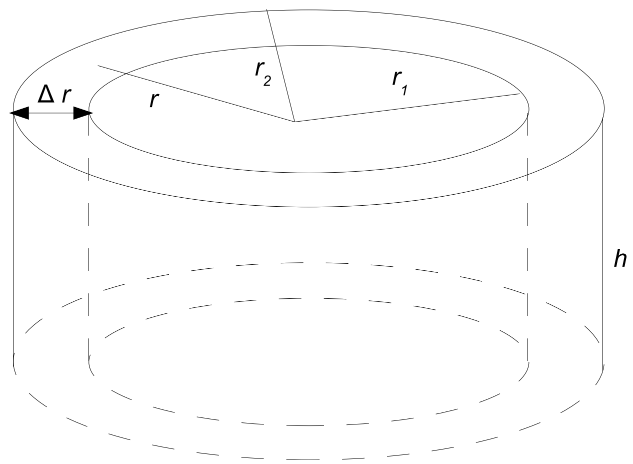

Fig.1

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi h(r_2^2 - r_1^2) = 2\pi h \frac{r_2 + r_1}{2}(r_2 - r_1)$$

Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is

$$V = 2\pi r h \Delta r$$

V =[circumference][height][thickness]
$V = 2\pi$[average radius][height][thickness]

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$

and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*)\Delta x_i$.

The approximation to the volume V of S is $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Thus, the volume of S is

$$V_Y = \lim_{\|P\|\to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i = 2\pi \int_a^b x f(x) dx$$

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$, $0 \le x \le 1$ about the y-axis.

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b $[f(x) \ge g(x)$ and $0 \le a < b]$ is

$$V_Y = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $x = 0$ about the y-axis, $x > 0$.

The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is

$$V_X = 2\pi \int_c^d y g(y) dy$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $x = 0$ about the x-axis.

The volume of the solid generated by rotating the region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$, about the x-axis, assuming that $g_2(y) \geq g_1(y)$ for all $c \leq x \leq d$, is

$$V_X = 2\pi \int_c^d y[g_2(y) - g_1(y)] dy$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.