Chapter 7. **Applications of integration**
Section 7.4 **Work**

Mechanical work is the amount of energy transferred by a force.

If an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion

$$F = ma = m\frac{d^2s}{dt^2}$$

In case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves

$$W = Fd, \text{ work=force } \times \text{ distance}$$

Mechanical units in the U.S. customary and SI metric systems

Unit	**U.S. customary system**	**SI metric system**
distance	ft	m
mass	$slug$	kg
force	lb	$N = kg \cdot m/sec^2$
work	$ft - lb$	$J = N \cdot m$
g(Earth)	$32ft/sec^2$	$9.81m/sec^2$

Example 1.

1. Find the work done in pushing a car a distance of 8 m while exerting a constant force of 900 N.

2. How much work is done by a weightlifter in raising a 60-kg barbell from the floor to the height of 2 m?

What happens if the force is variable?

Problem. The object moves along the x-axis in the positive direction from $x = a$ to $x = b$ and at each point x between a and b a force $f(x)$ acts on the object, where f is continuous function. Find the work done in moving the object from a to b.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$, and let x_i^* is in $[x_{i-1}, x_i]$. Then the force at x_i^* is $f(x_i^*)$. If $\|P\|$ is small, then Δx_i is small, and since f is continuous, the values of f do not change very much on $[x_{i-1}, x_i]$. In other words f is almost a constant on the interval and so work W_i that is done in moving the particle from x_{i-1} to x_i is $W_i \approx f(x_i^*)\Delta x_i$. We can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as $\|P\| \to 0$.

Therefore, we define the **work done in moving the object from** a **to** b as

$$W = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)dx$$

Example 2. When a particle is at a distance x meters from the origin, a force of $\cos(\pi x/3)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$.

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x

$$f(x) = kx$$

where k is a positive constant (the **spring constant**).

Example 3. Suppose that 2 J of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch it from 35 cm to 40 cm?

Example 4. A uniform cable hanging over the edge of a tall building is 40 ft long and weighs 60 lb. How much work is required to pull 10 ft of the cable to the top?

Example 5. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all the water out over the side?