Chapter 7. Applications of integration
Section 7.4 Work

Mechanical work is the amount of energy transferred by a force.

If an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion

$$F = ma = m\frac{d^2s}{dt^2}$$

In case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves

$$\boxed{W = Fd} \quad \text{work=force} \times \text{distance}$$

Mechanical units in the U.S. customary and SI metric systems

Unit	**U.S. customary system**	**SI metric system**
distance	ft	m
mass	$slug$	kg
force	lb	$N = kg \cdot m/sec^2$
work	$ft - lb$	$J = N \cdot m$
g(Earth)	$32ft/sec^2$	$9.81m/sec^2$

Example 1.

1. Find the work done in pushing a car a distance of 8 m while exerting a constant force of 900 N.

   $d=8$, $F=900$

   $\boxed{W = Fd = (900)(8) = 7200 \ (J)}$

2. How much work is done by a weightlifter in raising a 60-kg barbell from the floor to the height of 2 m?

   $d=2$

   $F = mg = 60(9.81)$

   $W = Fd = \boxed{2(60)(9.81) J}$

What happens if the force is variable?

Problem. The object moves along the x-axis in the positive direction from $x = a$ to $x = b$ and at each point x between a and b a force $f(x)$ acts on the object, where f is continuous function. Find the work done in moving the object from a to b.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$, and let x_i^* is in $[x_{i-1}, x_i]$. Then the force at x_i^* is $f(x_i^*)$. If $\|P\|$ is small, then Δx_i is small, and since f is continuous, the values of f do not change very much on $[x_{i-1}, x_i]$. In other words f is almost a constant on the interval and so work W_i that is done in moving the particle from x_{i-1} to x_i is $W_i \approx f(x_i^*)\Delta x_i$. We can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as $\|P\| \to 0$.

Therefore, we define the **work done in moving the object from** a **to** b as

$$W = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \boxed{\int_a^b f(x)dx = W}$$

Example 2. When a particle is at a distance x meters from the origin, a force of $\cos(\pi x/3)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$.

$f(x) = \cos\frac{\pi x}{3}$, $1 \le x \le 2$

$W = \int_1^2 f(x)\,dx = \int_1^2 \cos\frac{\pi x}{3}\,dx$

$u = \frac{\pi x}{3}$

$du = \frac{\pi}{3}dx$; $dx = \frac{3}{\pi}du$

$1 \to \frac{\pi}{3}(1) = \frac{\pi}{3}$

$2 \to \frac{\pi}{3}(2) = \frac{2\pi}{3}$

$= \frac{3}{\pi}\int_{\pi/3}^{2\pi/3} \cos u\,du$

$= \frac{3}{\pi}\sin u\Big]_{\pi/3}^{2\pi/3}$

$= \frac{3}{\pi}\left(\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}\right) = \boxed{0}$

$\boxed{\int \cos bx\,dx = \frac{1}{b}\sin bx + C}$

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x

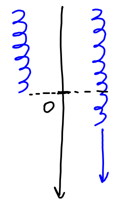
$$f(x) = kx$$

where k is a positive constant (the **spring constant**).

Example 3. Suppose that 2 J of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch it from 35 cm to 40 cm?

Convert cm into m. (inches into ft in US customary system).

natural lenght = 0



$30 \text{ cm} \mapsto 0$

$42 \text{ cm} \mapsto 42-30\,(\text{cm}) = 12\,(\text{cm}) = 0.12\ (\text{m})$

$F(x) = kx$, k is an unknown constant.

$2 = \int_0^{0.12} kx\,dx$ – equation for k.

$$2 = \frac{kx^2}{2}\Big]_0^{0.12}$$

$$2 = \frac{k}{2}(0.0144) \Rightarrow k = \frac{4}{0.0144} = \frac{40000}{144} = \frac{2500}{9}$$

$$k = \frac{2500}{9} \Rightarrow \boxed{f(x) = \frac{2500}{9}x}$$

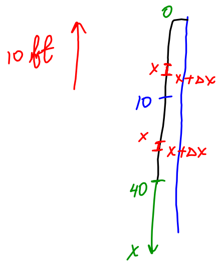
stretch from 35 cm to 40 cm

$35 \mapsto 35-30\,(\text{cm}) = 5\,(\text{cm}) = 0.05\ (\text{m})$

$40 \mapsto 40-30\,(\text{cm}) = 10\,(\text{cm}) = 0.1\ (\text{m})$

$$W = \int_{0.05}^{0.1} \frac{2500}{9}x\,dx = \frac{2500}{9}\,\frac{x^2}{2}\Big]_{0.05}^{0.1} = \boxed{\frac{2500}{2(9)}\left[0.01 - 0.0025\right]}$$

Example 4. A uniform cable hanging over the edge of a tall building is 40 ft long and weighs 60 lb. How much work is required to pull 10 ft of the cable to the top?



$0 \le x \le 40$

Case 1. Brake the cable into two parts.

- $0 \le x \le 10$.

Take a small part of the cable between x and $x+\Delta x$

[weight] = [density][length] = $\frac{60}{40}\Delta x = \frac{3}{2}\Delta x$

[distance travelled] = x

Work done by pulling up this part of the cable

$= (\text{weigth})(\text{distance}) = \frac{3}{2}\Delta x\, x$

$W_1 = \int_0^{10} \frac{3}{2} x\, dx$ — work done by pulling up the first 10 ft of the cable.

- $10 \le x \le 40$, small part between x and $x+\Delta x$

[weight] = $\frac{3}{2}\Delta x$

[distance] = 10

$W_2 = \int_{10}^{40} \frac{3}{2}(10)\, dx$ — work done by pulling up the rest of the cable.

$$\boxed{\text{Total work} = W_1 + W_2 = \int_0^{10} \frac{3}{2} x\, dx + \int_{10}^{40} \frac{3}{2}(10)\, dx}$$

Case 2. $W = \int_{10}^{40} \frac{3}{2} x\, dx$

. A cable 40 feet long weighing 6 pounds per foot is hanging off the side of a 50 foot tall building. At the bottom of the cable is a bucket of rocks weighing 100 pounds. How much work is required to pull 10 feet of the cable to the top of the building?

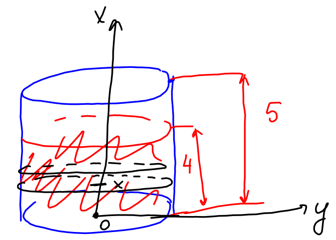
$$W = \underbrace{100(10)}_{\text{work done by pulling up the bucket of rocks}} + \underbrace{\int_{10}^{40} (6)\,x\,dx}_{\text{work done pulling up the rope.}}$$

. A cable 40 feet long weighing 6 pounds per foot is hanging off the side of a 50 foot tall building. At the bottom of the cable is a bucket of rocks weighing 100 pounds. How much work is required to pull ~~10 feet of~~ all the cable to the top of the building?

$$W = (100)(40) + \int_{0}^{40} (6)x\,dx$$

Example 5. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all the water out over the side? Use the fact that water weighs 62.5 lb/ft^3.

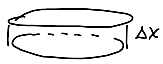
Take a "slice" of water between x and $x+\Delta x$, $0 \le x \le 4$.



$d = 24$ ft $\Rightarrow$ $r = 12$
$h = 5$ ft.

weigh of "slice" =?

distance traveled by "slice" =?



weigh of "slice" = (volume of "slice")(62.5)

volume $= \pi r^2 \Delta x$
$\pi (12)^2 \Delta x$

distance $= 5 - x$

$$\text{work done} = \int_0^4 (\text{weigh})(\text{distance})\,dx$$

$$= \int_0^4 (\pi)(12^2)(62.5)(5-x)\,dx$$

$$= 144\pi(62.5)\int_0^4 (5-x)\,dx$$

$$= 144\pi(62.5)\left(5x - \frac{x^2}{2}\right)\Big|_0^4$$

$$= \boxed{144\pi(62.5)(20-8)} \text{ (ft-lb)}$$