Section 7.5 **Average value of a function**

Let us try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b-a)/n$ and choose points x_i^* in successive subintervals. Then the average of the numbers $f(x_1^*)$, $f(x_2^*)$,...,$f(x_n^*)$, is

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{n}$$

Since $n = (b-a)\Delta x$,

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}(f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x)$$

The limiting value as $n \to \infty$ is

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

We define the **average value of** f on the interval $[a, b]$ as

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx}$$

Example 1. Find the average value of $f(x) = \sin^2 x \cos x$ on $[\pi/4, \pi/2]$.

$$f_{ave} = \frac{1}{\frac{\pi}{2} - \frac{\pi}{4}} \int_{\pi/4}^{\pi/2} \sin^2 x \cos x \, dx = \frac{4}{\pi}\int_{\pi/4}^{\pi/2} \sin^2 x \cos x \, dx$$

$u = \sin x$
$du = \cos x \, dx$
$\frac{\pi}{4} \to \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{2} \to \sin\frac{\pi}{2} = 1$

$$= \frac{4}{\pi}\int_{\frac{\sqrt{2}}{2}}^{1} u^2 du = \frac{4}{\pi} \left.\frac{u^3}{3}\right]_{\frac{\sqrt{2}}{2}}^{1} = \frac{4}{3\pi}\left(-\left(\frac{\sqrt{2}}{2}\right)^3 + 1\right)$$

$$= \frac{4}{3\pi}\left(-\frac{2\sqrt{2}}{8} + 1\right) = \boxed{\frac{4}{3\pi}\left(-\frac{\sqrt{2}}{4} + 1\right)}$$

Example 2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

$f_{ave} = 3, \quad f_{ave} = \frac{1}{b-0}\int_0^b f(x)dx$

$= \frac{1}{b}\int_0^b (2+6x-3x^2)dx = \frac{1}{b}\left(2x + \frac{6x^2}{2} - \frac{3x^3}{3}\right)_0^b$

$= \frac{1}{b}(2b + 3b^2 - b^3)$

$f_{ave} = 2 + 3b - b^2 = 3$

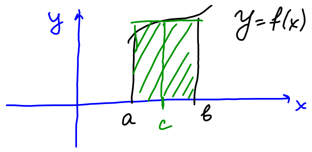
$b^2 - 3b + 1 = 0$

$\boxed{b_1 = \frac{3+\sqrt{9-4}}{2} = \frac{3+\sqrt{5}}{2}}$

$\boxed{b_2 = \frac{3-\sqrt{5}}{2}}$

Mean value theorem for integrals If f continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$\boxed{\int_a^b f(x)dx = f(c)(b-a) \quad a \le c \le b}$$



The geometric interpretation of this theorem for *positive* functions $f(x)$, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as a region under the graph of f from a to b.

Example 3. Find the average value of the function $f(x) = 4 - x^2$ on the interval $[0, 2]$. Find c $(0 \le c \le 2)$ such that $f_{ave} = f(c)$.

$f(x) = 4 - x^2$ on $[0,2]$

$f_{ave} = \frac{1}{2-0}\int_0^2 (4-x^2)dx = \frac{1}{2}\left(4x - \frac{x^3}{3}\right)_0^2$

$= \frac{1}{2}\left(8 - \frac{8}{3}\right) = \frac{1}{2}\cdot\frac{16}{3} = \boxed{\frac{8}{3} = f_{ave}}$

Find c such that $f(c) = \frac{8}{3}$

$4 - c^2 = \frac{8}{3}$

$c^2 = 4 - \frac{8}{3} = \frac{4}{3}$

$c = \pm\frac{2}{\sqrt{3}}$ but $0 \le c \le 2$

so $\boxed{c = \frac{2}{\sqrt{3}}}$