Chapter 8. **Techniques of integration**
Section 8.2 **Trigonometric integrals**

How to evaluate $\int \sin^m x \cos^n x \; dx$

(a) if $n = 2k+1$ (n is odd), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \; dx = \int \sin^m x (\cos^2 x)^k \cos x \; dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \; dx$$

Then substitute $u = \sin x$

(b) if $m = 2s+1$ (m is odd), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2s+1} x \cos^n x \; dx = \int (\sin^2 x)^s \cos^n x \sin x \; dx = \int (1 - \cos^2 x)^s \cos^n x \sin x \; dx$$

Then substitute $u = \cos x$

Example 1. Evaluate the integral.

1. $\int \cos^3 x \; dx$

2. $\int \sin^5 x \cos^4 x \; dx$

3. $\displaystyle\int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2} \; dx$

(c) if both m and n are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes useful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Example 2. Evaluate each of the following integrals

1. $\int\limits_0^{\pi/2} \sin^2 3x \; dx$

2. $\int \cos^4 x \; dx$

3. $\int\limits_0^{\pi/2} \sin^2 x \cos^2 x \; dx$

How to evaluate $\int \tan^m x \sec^n x \; dx$

(a) if the power of secant is even $n = 2k$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \; dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \; dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \; dx$$

Then substitute $u = \tan x$

Example 3.

1. $\int_0^{\pi/4} \tan^4 x \; \sec^2 x \; dx$

2. $\int \tan^2 x \; dx$

3. $\int \tan^4 x \; dx$

4. $\int \sec^4 x \; dx$

5. $\int \tan^3 x \; dx$

(b) if the power of tangent is odd ($m = 2s+1$), save a factor of $\tan x \sec x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2s+1} x \sec^n x \; dx = \int (\tan^2 x)^s \sec^m x \tan x \sec x \; dx = \int (\sec^2 x - 1)^s \sec^m x \tan x \sec x \; dx$$

Then substitute $u = \sec x$

Example 4.

1. $\int \tan^3 x \;\; \sec^3 x \; dx$

2. $\int \sec x \; dx$

3. $\int \sec^3 x \; dx$

Integrals of the form

$$\int \cot^m x \csc^n x \; dx$$

can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Example 5. Find

1. $\int \cot^4 x \csc^4 x \; dx$

2. $\int \cot^3 x \csc^2 x \; dx$

To evaluate the integrals

$$\text{(a)} \int \sin mx \;\; \cos nx \; dx$$

$$\text{(b)} \int \sin mx \ \sin nx \ dx$$

$$\text{(c)} \int \cos mx \ \cos nx \ dx$$

use the corresponding identity:

$$\text{(a)} \ \sin\alpha \ \cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

$$\text{(b)} \ \sin\alpha \ \sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\text{(c)} \ \cos\alpha \ \cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 6.

1. $\int \sin 5x \ \sin 2x \ dx$

2. $\int \sin 3x \ \cos x \ dx$

3. $\int \cos 3x \ \cos 4x \ dx$