

Test 1 Feb. 18, 7:30-9:30PM in HECC 110
over 6.4, 6.5, 7.1-7.5, 8.1, 8.2

Chapter 8. Techniques of integration
Section 8.2 Trigonometric integrals

How to evaluate $\int \sin^m x \cos^n x dx$

(a) if $n = 2k + 1$ (n is odd), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x (\cos^2 x)^k \cos x dx = \int \sin^m x (1 - \sin^2 x)^k \cos x dx$$

Then substitute $u = \sin x$

(b) if $m = 2s + 1$ (m is odd), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2s+1} x \cos^n x dx = \int (\sin^2 x)^s \cos^n x \sin x dx = \int (1 - \cos^2 x)^s \cos^n x \sin x dx$$

Then substitute $u = \cos x$

Example 1. Evaluate the integral.

$$1. \int \cos^3 x \, dx = \int \cos x \overbrace{(\cos^2 x)}^{1 - \sin^2 x} \, dx = \int \cos x (1 - \sin^2 x) \, dx \quad \left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right|$$

$$= \int (1 - u^2) \, du = u - \frac{u^3}{3} + C = \boxed{\sin x - \frac{\sin^3 x}{3} + C}$$

$$2. \int \sin^5 x \cos^4 x \, dx = \int \sin x (\sin^4 x) \cos^4 x \, dx$$

$$\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2$$

$$= \int \sin x (1 - \cos^2 x)^2 \cos^4 x \, dx \quad \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right|$$

$$= - \int \underbrace{(1 - u^2)^2}_{1 - 2u^2 + u^4} u^4 \, du = - \int (1 - 2u^2 + u^4) u^4 \, du = - \int (u^4 - 2u^6 + u^8) \, du$$

$$= - \frac{u^5}{5} + 2 \frac{u^7}{7} - \frac{u^9}{9} + C = \boxed{-\frac{\cos^5 x}{5} + 2 \frac{\cos^7 x}{7} - \frac{\cos^9 x}{9} + C}$$

$$3. \int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2} \, dx = \int \sin \frac{x}{2} \overbrace{\sin^2 \frac{x}{2}}^{1 - \cos^2 \frac{x}{2}} \cos^5 \frac{x}{2} \, dx$$

$$= \int \sin \frac{x}{2} (1 - \cos^2 \frac{x}{2}) \cos^5 \frac{x}{2} \, dx \quad \left| \begin{array}{l} u = \cos \frac{x}{2} \\ du = -\frac{1}{2} \sin \frac{x}{2} \, dx \\ \sin \frac{x}{2} \, dx = -2 \, du \end{array} \right|$$

$$= -2 \int (1 - u^2) u^5 \, du = -2 \int (u^5 - u^7) \, du = -2 \left(\frac{u^6}{6} - \frac{u^8}{8} \right) + C$$

$$= \frac{u^6}{3} - \frac{u^8}{4} + C = \boxed{\frac{\cos^6 \frac{x}{2}}{3} - \frac{\cos^8 \frac{x}{2}}{4} + C}$$

(c) if both m and n are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes useful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Example 2. Evaluate each of the following integrals

$$\begin{aligned} 1. \int_0^{\pi/2} \sin^2 3x \, dx &= \int_0^{\pi/2} \frac{1}{2}(1 - \cos 6x) \, dx = \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) \Big|_0^{\pi/2} \\ &= \frac{1}{2} \left(\frac{\pi}{2} - \frac{1}{6} \sin \frac{6\pi}{2} + \frac{1}{6} \sin 0 \right) \\ &= \boxed{\frac{\pi}{4}} \end{aligned}$$

$$\begin{aligned} 2. \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx = \int \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx = \int \frac{1}{4} (1 + 2\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right) \, dx \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x \right) \, dx \\ &= \boxed{\frac{1}{4} \left(\frac{3}{2}x + 2 \frac{1}{2} \sin 2x + \frac{1}{2} \frac{1}{4} \sin 4x \right) + C} \end{aligned}$$

$$\begin{aligned} 3. \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx &= \int_0^{\pi/2} \left(\frac{1}{2} \sin 2x \right)^2 \, dx = \frac{1}{4} \int_0^{\pi/2} \sin^2 2x \, dx = \frac{1}{4} \int_0^{\pi/2} \frac{1 - \cos 4x}{2} \, dx \\ &= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) \Big|_0^{\pi/2} = \boxed{\frac{\pi}{16}} \end{aligned}$$

How to evaluate $\int \tan^m x \sec^n x dx$

(a) if the **power of secant is even** $n = 2k$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx$$

Then substitute $u = \tan x$

Example 3.

$$1. \int_0^{\pi/4} \tan^4 x \sec^2 x dx \quad \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x dx \\ 0 \rightarrow \tan 0 = 0 \\ \frac{\pi}{4} \rightarrow \tan \frac{\pi}{4} = 1 \end{array} \right| = \int_0^1 u^4 du = \left. \frac{u^5}{5} \right|_0^1 = \frac{1}{5}$$

$$2. \int \tan^2 x dx$$

$$\tan^2 x = \sec^2 x - 1$$

$$= \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx = \tan x - x + C$$

$$3. \int \tan^4 x dx = \int (\tan^2 x)(\tan^2 x) dx = \int (\tan^2 x)(\sec^2 x - 1) dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$\left. \begin{array}{l} u = \tan x \\ du = \sec^2 x dx \end{array} \right|$

$$= \int u^2 du - (\tan x - x)$$

$$= \frac{u^3}{3} - (\tan x - x) + C$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

$$\begin{aligned}
 4. \int \sec^4 x \, dx &= \int \underbrace{(\sec^2 x)}_{\tan^2 x + 1} (\sec^2 x) \, dx = \int (\tan^2 x + 1) \sec^2 x \, dx \quad \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right. \\
 &= \int (u^2 + 1) \, du = \frac{u^3}{3} + u + C = \boxed{\frac{\tan^3 x}{3} + \tan x + C}
 \end{aligned}$$

$$\begin{aligned}
 5. \int \tan^3 x \, dx &= \int \tan x \underbrace{(\tan^2 x)}_{\sec^2 x - 1} \, dx = \int \tan x (\sec^2 x - 1) \, dx = \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\
 &= \int \tan x \sec^2 x \, dx - \int \frac{\sin x}{\cos x} \, dx \\
 &\quad \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right. \quad \left| \begin{array}{l} v = \cos x \\ dv = -\sin x \, dx \end{array} \right. \\
 &= \int u \, du + \int \frac{dv}{v} = \frac{u^2}{2} + \ln |v| + C \\
 &= \boxed{\frac{\tan^2 x}{2} + \ln |\cos x| + C}
 \end{aligned}$$

(b) if the **power of tangent is odd** ($m = 2s + 1$), save a factor of $\tan x \sec x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\tan^2 x = \sec^2 x - 1$$

$$\int \tan^{2s+1} x \sec^n x dx = \int (\tan^2 x)^s \sec^m x \tan x \sec x dx = \int (\sec^2 x - 1)^s \sec^m x \tan x \sec x dx$$

Then substitute $u = \sec x$

Example 4.

$$\begin{aligned} 1. \int \tan^3 x \sec^3 x dx &= \int (\tan x \sec x) \overbrace{\tan^2 x \sec^2 x}^{\sec^2 x - 1} dx = \int (\tan x \sec x) (\sec^2 x - 1) \sec^2 x dx \quad \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x dx \end{array} \right. \\ &= \int (u^2 - 1)u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C} \end{aligned}$$

$$\begin{aligned} 2. \int \sec x dx &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx \quad \left| \begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x) dx \end{array} \right. = \int \frac{du}{u} \\ &= \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C} \end{aligned}$$

$$\int u v' dx = uv - \int u' v dx$$

$$3. \int \sec^3 x dx = \int (\sec x)(\sec^2 x) dx = \left. \begin{array}{l} u = \sec x \\ v' = \sec^2 x \end{array} \right| \begin{array}{l} u' = \sec x \tan x \\ v = \tan x \end{array}$$

$$= \sec x \tan x - \int (\sec x \tan x)(\tan x) dx = \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x dx$$

$$2 \int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x|$$

$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$

Integrals of the form

$$\int \cot^m x \csc^n x dx$$

$$(\cot x)' = -\csc^2 x$$

$$(\csc x)' = -\cot x \csc x$$

can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Example 5. Find

$$1. \int \cot^4 x \csc^4 x dx = \int \cot^4 x \underbrace{(\csc^2 x)}_{1 + \cot^2 x} (\csc^2 x) dx = \int \cot^4 x (1 + \cot^2 x) \csc^2 x dx \quad \left| \begin{array}{l} u = \cot x \\ du = -\csc^2 x dx \end{array} \right|$$

$$= -\int u^4 (1 + u^2) du = -\int (u^4 + u^6) du = -\left(\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\left(\frac{\cot^5 x}{5} + \frac{\cot^7 x}{7}\right) + C$$

$$2. \int \cot^3 x \csc^2 x dx = \int (\cot x \csc x) \overbrace{\cot^2 x \csc x}^{\csc^2 x - 1} dx = \int (\cot x \csc x) (\csc^2 x - 1) \csc x dx \quad \left| \begin{array}{l} u = \csc x \\ du = -\csc x \cot x dx \end{array} \right|$$

$$= -\int (u^2 - 1) u du = -\int (u^3 - u) du = -\left(\frac{u^4}{4} - \frac{u^2}{2}\right) + C$$

$$= -\left(\frac{\csc^4 x}{4} - \frac{\csc^2 x}{2}\right) + C$$

To evaluate the integrals

$$(a) \int \sin mx \cos nx \, dx$$

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$$(b) \int \sin mx \sin nx \, dx$$

$$(c) \int \cos mx \cos nx \, dx$$

use the corresponding identity:

$$(a) \sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$(b) \sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$(c) \cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

Example 6.

1. $\int \sin 5x \sin 2x \, dx$

$$\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$= \int \frac{1}{2} (\cos(5x - 2x) - \cos(5x + 2x)) \, dx$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C = \frac{1}{2} \int (\cos 3x - \cos 7x) \, dx = \frac{1}{2} \left(\frac{1}{3} \sin 3x - \frac{1}{7} \sin 7x \right) + C$$

2. $\int \sin 3x \cos x \, dx$

$$\sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$= \int \frac{1}{2} (\sin(3x + x) + \sin(3x - x)) \, dx = \frac{1}{2} \int (\sin 4x + \sin 2x) \, dx$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C = -\frac{1}{2} \left(\frac{1}{4} \cos 4x + \frac{1}{2} \cos 2x \right) + C$$

3. $\int \cos 3x \cos 4x \, dx$

$$\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$= \int \frac{1}{2} (\cos(3x - 4x) + \cos(3x + 4x)) \, dx = \frac{1}{2} \int [\overbrace{\cos(-x)}^{\cos x} + \cos 7x] \, dx$$

$$= \frac{1}{2} \int [\cos x + \cos 7x] \, dx = \frac{1}{2} \left(\sin x + \frac{1}{7} \sin 7x \right) + C$$