Chapter 8. **Techniques of integration**
Section 8.3 **Trigonometric substitution**

Assume that g is one-to-one function (g^{-1} exists). Then

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

$x = g(t)$
$dx = g'(t)\,dt$

This kind of substitution is called *inverse substitution*.

Table of trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin t, -\pi/2 \le t \le \pi/2$	$1-\sin^2 t = \cos^2 t$
$\sqrt{a^2+x^2}$	$x = a\tan t, -\pi/2 < t < \pi/2$	$1+\tan^2 t = \sec^2 t$
$\sqrt{x^2-a^2}$	$x = a\sec t, 0 \le t \le \pi/2$ or $\pi \le t \le 3\pi/2$	$\sec^2 t - 1 = \tan^2 t$

Example Find

(a) $\int \underbrace{x}_{2\sin t}\underbrace{\sqrt{4-x^2}}_{2\cos t}\underbrace{dx}_{2\cos t dt}$

$x = 2\sin t$
$dx = 2\cos t\,dt$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2 t}$
$= \sqrt{4\underbrace{(1-\sin^2 t)}_{\cos^2 t}}$
$= \sqrt{4\cos^2 t}$

$\sqrt{4-x^2} = 2\cos t$

$\cos t = \dfrac{\sqrt{4-x^2}}{2}$

$= \int \underbrace{2\sin t}_{x}\ \underbrace{2\cos t}_{\sqrt{4-x^2}}\ \underbrace{2\cos t\,dt}_{dx}$

$= 8\int \sin t\cos^2 t\,dt$

$u = \cos t$
$du = -\sin t\,dt$

$= -8\int u^2 du = -8\dfrac{u^3}{3} + C = -\dfrac{8}{3}\cos^3 t + C$

$= -\dfrac{8}{3}\dfrac{(\sqrt{4-x^2})^3}{(2)^3} + C = \boxed{-\dfrac{1}{3}(\sqrt{4-x^2})^3 + C}$

(b) $\int \frac{x^3}{\sqrt{x^2+4}}dx$

$x^2+4 = x^2+2^2$

$x = 2\tan t$

$dx = 2\sec^2 t\, dt$

$\sqrt{x^2+4} = \sqrt{4\tan^2 t+4}$

$= \sqrt{4(\tan^2 t+1)}$

$= \sqrt{4\sec^2 t}$

$\sqrt{x^2+4} = 2\sec t$

$\sec t = \frac{\sqrt{x^2+4}}{2}$

$= \int \frac{\overset{x^3 \,/\!/}{(2\tan t)^3}}{\underset{/\!/\, \sqrt{x^2+4}}{2\sec t}} \overset{dx \,/\!/}{2\sec^2 t\, dt}$

$= \int \frac{8\tan^3 t}{\cancel{2\sec t}} \cancel{2}\sec^{\cancel{2}} t\, dt$

$= 8\int \tan^3 t \sec t\, dt = 8\int (\tan t \sec t)\overbrace{\tan^2 t}^{\sec^2 t - 1}\, dt$

$= 8\int (\tan t \sec t)(\sec^2 t - 1)\, dt$ $\quad\left|\begin{array}{l} u = \sec t \\ du = \sec t \tan t\, dt \end{array}\right|$

$= 8\int (u^2-1)\,du = 8\left(\frac{u^3}{3} - u\right) + C = 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C$

$= \boxed{8\left(\frac{(x^2+4)^{3/2}}{3(8)} - \frac{\sqrt{x^2+4}}{2}\right) + C}$

(c) $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{x^2\sqrt{(4x)^2-3^2}}$

$4x = 3\sec t$ or $x = \frac{3}{4}\sec t$

$dx = \frac{3}{4}\sec t \tan t \, dt$

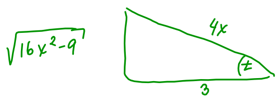
$\sqrt{(4x)^2-3^2} = \sqrt{(3\sec t)^2 - 9}$

$= \sqrt{9\sec^2 t - 9}$

$= \sqrt{9(\sec^2 t - 1)}$

$= \sqrt{9\tan^2 t}$

$\sqrt{16x^2-9} = 3\tan t$



$x = \frac{3}{4}\sec t$

$\sec t = \frac{4x}{3} = \frac{1}{\cos t}$

$\cos t = \frac{3}{4x}$

$\boxed{\sin t = \frac{\sqrt{16x^2-9}}{4x}}$

$= \displaystyle\int \frac{\cancel{\frac{3}{4}}\,\cancel{\sec t}\,\cancel{\tan t}\,dt}{\frac{9}{16}\sec^{\cancel{2}} t\,(\cancel{3}\cancel{\tan t})} = \frac{1}{4}\cdot\frac{16}{9}\int \frac{dt}{\sec t} = \frac{4}{9}\int \cos t\,dt = \frac{4}{9}\sin t + C$

$= \frac{\cancel{4}}{9}\,\frac{\sqrt{16x^2-9}}{\cancel{4}x} + C = \boxed{\frac{\sqrt{16x^2-9}}{9x} + C}$

(d) $\int \frac{dx}{\sqrt{x^2+4x+8}}$

Complete the square: $\underbrace{x^2+4x}_{}+8 = (x^2+4x+4)+4 = (x+2)^2+4 = (x+2)^2+2^2$

$(x+2)^2 = x^2+4x+4$

$= \int \frac{dx}{\sqrt{(x+2)^2+2^2}} =$

$\tan t = \frac{x+2}{2}$

$x+2 = 2\tan t$ or $x = 2\tan t - 2$

$dx = 2\sec^2 t\, dt$

$\sqrt{(x+2)^2+2^2} = \sqrt{(2\tan t)^2+2^2}$

$= \sqrt{4\tan^2 t+4} = \sqrt{4(\tan^2 t+1)}$

$= \sqrt{4\sec^2 t}$

$\sqrt{(x+2)^2+4} = 2\sec t \Rightarrow \sec t = \frac{\sqrt{(x+2)^2+4}}{2}$

$= \int \frac{2\sec^2 t\, dt}{2\sec t}$

$= \int \sec t\, dt$

$= \ln|\sec t + \tan t| + C$

$= \boxed{\ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right| + C}$

(e) $\int e^t\sqrt{9-e^{2t}}\,dt$

$\left| \begin{array}{l} u = e^t \\ e^{2t} = u^2 \\ du = e^t\,dt \end{array} \right| = \int \sqrt{9-u^2}\,du$

$u = 3\sin x \Rightarrow \sin x = \frac{u}{3} \Rightarrow x = \arcsin\left(\frac{u}{3}\right)$

$du = 3\cos x\,dx$

$\sqrt{9-u^2} = \sqrt{9-(3\sin x)^2}$

$= \sqrt{9-9\sin^2 x} = \sqrt{9(1-\sin^2 x)}$

$= \sqrt{9\cos^2 x}$

$\sqrt{9-u^2} = 3\cos x \Rightarrow \cos x = \frac{\sqrt{9-u^2}}{3}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \int 3\cos x \; 3\cos x\,dx = 9\int \cos^2 x\,dx$

$= \frac{9}{2}\int (1+\cos 2x)\,dx = \frac{9}{2}\left(\underset{\arcsin\left(\frac{u}{3}\right)}{x} + \frac{1}{2}\sin 2x\right) + C$

$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

$\frac{1}{2}\sin 2x = \frac{1}{\cancel{2}}\cancel{2}\sin x\cos x = \underset{\sin x}{\left(\frac{u}{3}\right)}\underset{\cos x}{\left(\frac{\sqrt{9-u^2}}{3}\right)} = \frac{u\sqrt{9-u^2}}{9}$

$= \frac{9}{2}\left(\arcsin\left(\frac{u}{3}\right) + \frac{u\sqrt{9-u^2}}{9}\right) + C$

$= \boxed{\frac{9}{2}\left(\arcsin\left(\frac{e^t}{3}\right) + \frac{e^t\sqrt{9-e^{2t}}}{9}\right) + C}$