Chapter 8. **Techniques of integration**
Section 8.4 **Integration of rational functions by partial fractions**

In this section we show how to integrate any rational function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0x^m + b_1x^{m-1} + ... + b_m$ by expressing it as a sum of *partial fractions*, that we know how to integrate.

STEP 1. If f is improper ($n \geq m$), then we must divide P into Q by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax + b$ and *irreducible quadratic factors* (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

STEP 3. Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Four cases occur.

CASE I. $Q(x)$ **is a product of distinct linear factors.**

$$Q(x) = (a_1x + b_1)(a_2x + b_2)...(a_mx + b_m)$$

where no factor is repeated. Then there exist constants A_1, A_2,..., A_m such that

$$f(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_m}{a_mx + b_m}$$

Once the constants A_1, A_2,... , A_m are determined, the evaluation of $\frac{R(x)}{Q(x)}$ becomes a routine problem. The next example will illustrate one method for finding these constants.

Example 1. Evaluate $\displaystyle\int_{2}^{4} \frac{4x-1}{x^2+x-2}dx$

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $a_1x + b_1$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + ... + \frac{A_r}{(a_1x + b_1)^r}$$

Example 2. Evaluate $\int \frac{5x^2 + 6x + 9}{(x+1)^2(x-3)^2} dx$

CASE III $Q(x)$ **contains irreducible quadratic factors none of which is repeated.**

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the corresponding fraction is

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

The term $\dfrac{Ax + B}{ax^2 + bx + c}$ can be integrating by completing the square in the denominator.

Example 3. Find $\displaystyle\int \frac{3x^3 - x^2 + 6x - 4}{(x^2 + 1)(x^2 + 2)} dx$

CASE IV $Q(x)$ **contains a repeated irreducible factor.**

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction $\dfrac{Ax+B}{ax^2+bx+c}$, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+...+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of these terms can be integrated by completing the square and making the tangent substitution.

Example 4. Write out the form of the partial fraction decomposition of the function

$$\frac{x-3}{(x^2+x+1)^2(x^2+2x+4)^2}.$$

Do not determine the numerical values for the coefficients.