Quiz tomorrow over 8.3

Section 8.4 **Integration of rational functions by partial fractions**

In this section we show how to integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0x^m + b_1x^{m-1} + ... + b_m$ by expressing it as a sum of *partial fractions*, that we know how to integrate.

STEP 1. If f is improper ($n \geq m$), then we must divide P into Q by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax + b$ and *irreducible quadratic factors* (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

STEP 3. Express the proper rational function $\dfrac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Four cases occur.

CASE I. $Q(x)$ **is a product of distinct linear factors.**

$a_1x+b_1 \rightarrow \dfrac{A_1}{a_1x+b_1}$

$$Q(x) = (a_1x + b_1)(a_2x + b_2)...(a_mx + b_m)$$

where no factor is repeated. Then there exist constants A_1, A_2,..., A_m such that

A_1 is an unknown constant

$$f(x) = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_m}{a_mx+b_m}$$

Once the constants A_1, A_2,... , A_m are determined, the evaluation of $\dfrac{R(x)}{Q(x)}$ becomes a routine problem. The next example will illustrate one method for finding these constants.

Example 1. Evaluate $\int_2^4 \frac{4x-1}{x^2+x-2}dx$

proper – $\frac{4x-1}{x^2+x-2}$

$x^2+x-2=(x+2)(x-1)$ two nonrepeated linear factors

Partial fraction decomposition: $\frac{4x-1}{x^2+x-2}=\frac{4x-1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$

A and B are unknown constants.

$$\frac{4x-1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$$

$$\frac{4x-1}{\cancel{(x+2)(x-1)}}=\frac{A(x-1)+B(x+2)}{\cancel{(x+2)(x-1)}}$$

$$4x-1=A(x-1)+B(x+2)$$

$x=1$: $4(1)-1=A(0)+B(1+2)$

$3=3B \Rightarrow \boxed{B=1}$

$x=-2$: $4(-2)-1=A(-2-1)+B(-2+2)$

$-9=-3A \Rightarrow \boxed{A=3}$

$$\frac{4x-1}{(x+2)(x-1)}=\frac{3}{x+2}+\frac{1}{x-1}$$

$4x-1=Ax-A+Bx+2B$

$4x-1=(A+B)x+(2B-A)$

Equate coefficients to powers of x:

$x: \ 4=A+B \Rightarrow A=4-B$

$1: \ -1=2B-A$

$-1=2B-(4-B)$

$-1=2B-4+B$

$3=3B \Rightarrow \boxed{B=1}$

$A=4-B=4-1=\boxed{3=A}$

$$\int_2^4\frac{(4x-1)\,dx}{(x+2)(x-1)}=\int_2^4\left[\frac{3}{x+2}+\frac{1}{x-1}\right]dx=\Big[3\ln|x+2|+\ln|x-1|\Big]_2^4$$

$$=3\ln 6+\ln 3-3\ln 4+\ln 1^{\to 0}$$

$$=3(\ln 6-\ln 4)+\ln 3=3\ln\frac{6}{4}+\ln 3$$

$$=3\ln\frac{3}{2}+\ln 3$$

$$=\ln\left(\frac{3}{2}\right)^3+\ln 3=\ln\frac{27}{8}+\ln 3=\ln\left(\frac{27}{8}(3)\right)$$

$$=\boxed{\ln\frac{81}{8}}$$

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated.**
Suppose the first linear factor a_1x+b_1 is repeated r times; that is, $(a_1x+b_1)^r$ occurs in factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$, we would use

$$(a_1x+b_1)^r \longrightarrow \overbrace{\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\ldots+\frac{A_r}{(a_1x+b_1)^r}}^{r\text{ fractions.}}$$

Example 2. Evaluate $\displaystyle\int \frac{5x^2+6x+9}{(x+1)^2(x-3)^2}dx$

Proper $\displaystyle\frac{5x^2+6x+9}{(x+1)^2(x-3)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-3}+\frac{D}{(x-3)^2}$

$$\frac{5x^2+6x+9}{\cancel{(x+1)^2(x-3)^2}}=\frac{A(x+1)(x-3)^2+B(x-3)^2+C(x-3)(x+1)^2+D(x+1)^2}{\cancel{(x+1)^2(x-3)^2}}$$

$5x^2+6x+9=A(x+1)(x-3)^2+B(x-3)^2+C(x-3)(x+1)^2+D(x+1)^2$

$x=-1$: $5(-1)^2+6(-1)+9=0+B(-1-3)^2+0+0$

$8=16B \Rightarrow B=\boxed{\frac{1}{2}}$

$x=3$: $5(3^2)+6(3)+9=0+0+0+D(3+1)^2$

$45+18+9=16D$

$72=16D \Rightarrow D=\frac{72}{16}=\boxed{\frac{9}{2}=D}$

$x=0$: $9=A(1)(-3)^2+B(-3)^2+C(-3)(1)^2+D(1)^2$

$9=9A+9B-3C+D \quad (B=1/2,\ D=9/2)$

$9=9A+9\left(\frac{1}{2}\right)-3C+\frac{9}{2}$

$\frac{\cancel{9}}{3}=\frac{9A-3C}{3}+\cancel{9}$ $\boxed{C=3A}$

$x=1$: $5+6+9=A(2)(1-3)^2+B(1-3)^2+C(1-3)(1+1)^2+D(1+1)^2$

$\frac{20}{4}=\frac{8A+4B-8C+4D}{4}$

$5=2A+B-2C+D$ ($B \to 1/2$, $D \to 9/2$)

$5=2A+\frac{1}{2}-2C+\frac{9}{2}$

$5=2A-2C+5 \Rightarrow \boxed{A=C}$

System for A and C: $\begin{cases}C=3A\\C=A\end{cases}$ $\boxed{A=C=0}$

$$\frac{5x^2+6x+9}{(x+1)^2(x-3)^2}=\frac{\cancel{A}^{\,0}}{x+1}+\frac{\cancel{B}^{\,1/2}}{(x+1)^2}+\frac{\cancel{C}^{\,0}}{x-3}+\frac{\cancel{D}^{\,9/2}}{(x-3)^2}$$

$$\int\frac{5x^2+6x+9}{(x+1)^2(x-3)^2}dx=\int\left(\frac{1}{2}\,\frac{1}{(x+1)^2}+\frac{9}{2}\,\frac{1}{(x-3)^2}\right)dx \qquad \boxed{\int\frac{dx}{(x+a)^2}=-\frac{1}{x+a}+C}$$

$$=\boxed{-\frac{1}{2}\,\frac{1}{x+1}-\frac{9}{2}\,\frac{1}{x-3}+C}$$

CASE III $Q(x)$ **contains irreducible quadratic factors none of which is repeated.**
If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then the corresponding fraction is

$$ax^2+bx+c \;(b^2-4ac<0) \longrightarrow \frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants to be determined.

The term $\dfrac{Ax+B}{ax^2+bx+c}$ can be integrating by completing the square in the denominator.

Example 3. Find $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}dx$

$$\frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)} = \frac{\overset{3}{A}x+\overset{-3}{B}}{x^2+1} + \frac{\overset{0}{C}x+\overset{2}{D}}{x^2+2}$$

$$\frac{3x^3-x^2+6x-4}{\cancel{(x^2+1)(x^2+2)}} = \frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{\cancel{(x^2+1)(x^2+2)}}$$

$$3x^3-x^2+6x-4 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1)$$
$$= Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D$$
$$3x^3-x^2+6x-4 = x^3(A+C)+x^2(B+D)+x(2A+C)+(2B+D)$$

$$\left.\begin{aligned} x^3:&\quad 3 = A+C\\ x^2:&\quad -1 = B+D\\ x:&\quad 6 = 2A+C\\ 1:&\quad -4 = 2B+D \end{aligned}\right\}$$

$$\begin{cases} A+C=3\\ 2A+C=6\end{cases}$$
$C=3-A$
$2A+(3-A)=6$
$\boxed{A=3}\ \boxed{C=0}$

$$\begin{cases} B+D=-1\\ 2B+D=-4\end{cases}$$
$D=-1-B$
$2B+(-1-B)=-4$
$\boxed{B=-3}\ \boxed{D=2}$

$$\int\frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}dx = \int\left(\frac{3x-3}{x^2+1}+\frac{2}{x^2+2}\right)dx$$

$$=\int\left(\frac{3x}{x^2+1}-\frac{3}{x^2+1}+\frac{2}{x^2+2}\right)dx$$

$$=\frac{3}{2}\int\frac{2x}{x^2+1}dx - 3\int\frac{dx}{x^2+1}+2\int\frac{dx}{x^2+2} = \frac{3}{2}\int\frac{du}{u} - 3\arctan x + 2\int\frac{dx}{x^2+(\sqrt{2})^2}$$

$u=x^2+1$
$du=2x\,dx$

$$= \frac{3}{2}\ln|u| - 3\arctan x + \frac{2}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$$

$$= \boxed{\frac{3}{2}\ln|x^2+1| - 3\arctan x + \sqrt{2}\arctan\frac{x}{\sqrt{2}}+C}$$

CASE IV $Q(x)$ **contains a repeated irreducible factor.**

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction $\dfrac{Ax+B}{ax^2+bx+c}$, the sum

$$(ax^2+bx+c)^r \longrightarrow \frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of these terms can be integrated by completing the square and making the tangent substitution.

Example 4. Write out the form of the partial fraction decomposition of the function

$$\frac{x-3}{(x^2+x+1)^2(x^2+2x+4)^2}.$$

Do not determine the numerical values for the coefficients.

$$\boxed{\frac{x-3}{(x^2+x+1)^2(x^2+2x+4)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}+\frac{Ex+F}{x^2+2x+4}+\frac{Gx+H}{(x^2+2x+4)^2}}$$