Section 8.9 Improper integrals

In this section we extend the conception of a definite integral to the case where the interval is infinite and also to the case where integrand is unbounded.

1. **Definition of an improper integral of type 1 (infinite intervals)**

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\boxed{\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx}$$

provided this limit exists (as a finite number)

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\boxed{\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx}$$

provided this limit exists (as a finite number)

The improper integrals in (a) and (b) are called convergent if the limit exist and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\boxed{\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx}$$

where a is any real number

Example 1. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}dx$ convergent?

negative

$\lim t^{-2} = 0.$

$\boxed{\lim_{t\to\infty} \frac{1}{t^2} = 0}$

1. $p>1$

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \frac{1}{-p+1}\underbrace{\lim_{t\to\infty} t^{-p+1}}_{\to 0} - \frac{1}{-p+1} = -\frac{1}{-p+1} = \frac{1}{p-1}$$ convergent, if $p>1$

negative

if $p>1$, then $-p<-1$ and $-p+1<0$

2. $p=1$

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty}(\ln|t| - \underbrace{\ln 1}_{0}) = \lim_{t\to\infty}\ln|t| = \infty$$

divergent, if $p=1$

3. $p<1$

positive

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\frac{t^{(-p+1)}}{-p+1} - \frac{1}{-p+1} = \infty$$

$p<1 \Rightarrow 1-p>0$ divergent, if $p<1$

$$\boxed{\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } p\leq 1\end{cases}}$$

Example 2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

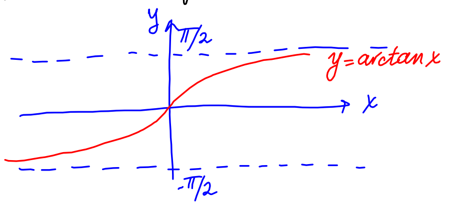
(a) $\displaystyle\int_2^\infty \frac{dx}{\sqrt[3]{x+3}}$ $\qquad \frac{1}{x^{1/3}} \qquad 1/3 < 1 \qquad \boxed{\text{divergent}}$

(b) $\displaystyle\int_{-\infty}^3 \frac{dx}{x^2+9} = \lim_{t\to-\infty} \int_t^3 \frac{dx}{x^2+3^2} = \lim_{t\to-\infty} \frac{1}{3}\arctan\frac{x}{3}\Big]_t^3 = \frac{1}{3}\left(\arctan\frac{3}{3} - \lim_{t\to-\infty}\arctan\frac{t}{3}\right)$

$2>1$ should be convergent

$\displaystyle = \frac{1}{3}\left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = \boxed{\frac{\pi}{4}}$

$\displaystyle\lim_{t\to-\infty} \arctan t = -\frac{\pi}{2}$



(c) $\displaystyle\int_{-\infty}^{\infty} (2x^2 + x - 1)\,dx = \boxed{\text{divergent}}$

(d) $\displaystyle\int_0^\infty \frac{1}{(x+2)(x+3)}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)(x+3)}\,dx = \lim_{t\to\infty} \int_0^t \left[\frac{1}{x+2} - \frac{1}{x+3}\right] dx$

convergent

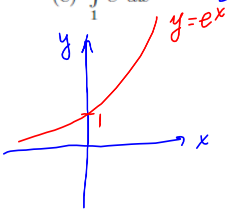
$$\boxed{\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3} = \frac{x+3-(x+2)}{(x+2)(x+3)} = \frac{1}{(x+2)(x+3)}}$$

$$= \lim_{t\to\infty}\left(\ln|x+2| - \ln|x+3|\right)\Big|_0^t = \lim_{t\to\infty} \ln\frac{|x+2|}{|x+3|}\Big]_0^t = \lim_{t\to\infty}\ln\left|\frac{t+2}{t+3}\right| - \ln\frac{2}{3}$$

$$= \ln\left|\lim_{t\to\infty}\frac{t+2}{t+3}\right| - \ln\frac{2}{3} = \ln 1 - \ln\frac{2}{3} = \boxed{-\ln\frac{2}{3}}$$

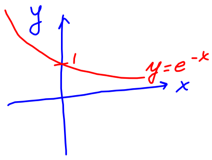
$$\lim_{t\to\infty}\frac{t+2}{t+3} = \lim_{t\to\infty}\frac{\cancel{t}(1+\frac{2}{t})}{\cancel{t}(1+3/t)} = \lim_{t\to\infty}\frac{1+\cancel{\frac{2}{t}}^{\,0}}{1+\cancel{3/t}^{\,0}} = 1$$

(e) $\int_1^\infty e^x dx = \lim_{t\to\infty} \int_1^t e^x dx = \lim_{t\to\infty} (e^t - e^1) = \lim_{t\to\infty} e^t - e = \infty$ $\boxed{\text{divergent}}$



$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$



$\lim_{x\to\infty} e^{-x} = 0$

$\lim_{x\to-\infty} e^{-x} = \infty$

(f) $\int_{-\infty}^1 e^x dx = \lim_{t\to-\infty} \int_t^1 e^x dx = \lim_{t\to-\infty} e^x\Big]_t^1 = \lim_{t\to-\infty} (e - e^t) = e - \lim_{t\to-\infty} e^t = \boxed{e}$

convergent

(g) $\int_{-\infty}^\infty e^x dx = \underbrace{\int_{-\infty}^1 e^x dx}_{\text{convergent}} + \overbrace{\int_1^\infty e^x dx}^{\text{divergent}} = \infty$ $\boxed{\text{divergent}}$

$\boxed{\text{Convergent}} + \boxed{\text{divergent}} = \boxed{\text{divergent}}$

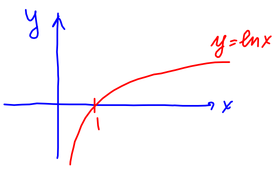
(h) $\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^2}$

$\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ e \to \ln e = 1 \\ t \to \ln t \end{array} \right|$

$= \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^2} = \lim_{t\to\infty} \left[-\frac{1}{u}\right]_1^{\ln t}$

$= \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{1}\right)$

$= \overset{0}{\cancel{-\lim_{t\to\infty} \frac{1}{\ln t}}} + 1 = \boxed{1}$



$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to0^+} \ln x = -\infty$

2. **Definition of an improper integral of type 2 (discontinuous integrands)**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\boxed{\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx}$$

if this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\boxed{\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx}$$

if this limit exists (as a finite number)

The improper integrals in (a) and (b) are called convergent if the limit exist and divergent if the limit does not exist.

(c) If f has discontinuity at c $(a < c < b)$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$$

Example 3. For what values of p is the integral $\int_0^1 \frac{1}{x^p}dx$ convergent?

$\frac{1}{x^p}$ discontinuous @ $x=0$.

1) $p>1 \Rightarrow -p+1<0$

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^p}dx = \lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Big]_t^1 = \frac{1}{-p+1} - \lim_{t\to 0^+} \frac{t^{-p+1}}{-p+1} = \infty$$

negative

$= \infty$

$\boxed{\text{divergent, if } p>1}$

2) $p=1$

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+} \ln|x|\Big]_t^1 = \lim_{t\to 0^+} (\ln 1 - \ln|t|) = -\lim_{t\to 0^+} \ln|t| = \infty$$

$\boxed{\text{divergent, if } p=1}$

3) $p<1 \Rightarrow -p+1>0$

positive

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0^+} \int_t^1 x^{-p}dx = \lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Big]_t^1 = \frac{1}{-p+1} - \lim_{t\to 0^+} \frac{t^{-p+1}}{-p+1} = \frac{1}{-p+1}$$

$\boxed{\text{convergent, if } p<1}$

$$\boxed{\int_0^1 \frac{1}{x^p}dx = \begin{cases} \infty, & \text{if } p \geq 1 \\ \frac{1}{-p+1}, & \text{if } p<1 \end{cases}}$$

Example 4. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_{-3}^{0}\frac{dx}{\sqrt{x+3}} = \lim_{t\to -3^+}\int_t^0 \frac{dx}{\sqrt{x+3}} = \lim_{t\to -3^+}\int_t^0 (x+3)^{-1/2}\,dx = \lim_{t\to -3^+}\frac{(x+3)^{1/2}}{1/2}\Big]_t^0$

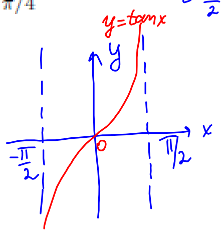
$p=\frac{1}{2}<1$ convergent.

$\displaystyle = 2\left(3^{1/2} - \lim_{t\to -3^+}\overbrace{(t+3)^{1/2}}^{0}\right) = \boxed{2\sqrt{3}}$

2. $\displaystyle\int_{0}^{3}\frac{1}{x\sqrt{x}}dx \qquad p=\frac{3}{2}>1 \qquad \boxed{\text{divergent}}$

3. $\displaystyle\int_{\pi/4}^{\pi/2}\sec^2 x\,dx = \lim_{t\to\frac{\pi}{2}^-}\int_{\pi/4}^{t}\sec^2 x\,dx = \lim_{t\to\frac{\pi}{2}^-}\tan x\Big]_{\pi/4}^{t} = \lim_{t\to\frac{\pi}{2}^-}\overbrace{\tan t}^{\infty} - \tan\frac{\pi}{4} = \infty$

$\boxed{\text{divergent}}$



integrate by parts

$$4. \int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \left| \begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix} \right| = \lim_{t\to 0^+} \left(x\ln x \Big]_t^1 - \int_t^1 \frac{1}{x}\,x\,dx \right)$$

$$= \lim_{t\to 0^+} \left(-t\ln t - x \Big]_t^1 \right) = \lim_{t\to 0^+} \left(-t\ln t - 1 + t \right)$$

0·∞ d'Hospitale Rule

$$= -\lim_{t\to 0^+} t\ln t - 1 + \cancel{\lim_{t\to 0^+}} \overset{0}{t} = -1 - \lim_{t\to 0^+} t\ln t$$

$$= -1 - \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -1 + \cancel{\lim_{t\to 0^+}} \overset{0}{t} = \boxed{-1}$$

$\int_a^\infty f(x)dx \geq \int_a^\infty g(x)dx$

Comparison theorem. Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example 5. Use the Comparison Theorem to determine whether $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent or divergent.

$0 \leq \sin^2 x \leq 1 \implies 0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$

$\int_1^\infty \frac{\sin^2 x}{x^2} dx$ compare with $\underbrace{\int_1^\infty \frac{1}{x^2} dx}_{\text{conv.}}$

By Comparison Thm. Part a, $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is $\boxed{\text{convergent}}$