

Section 9.3 Arc length

Let's a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

Assume that C is smooth ($x'(t)$ and $y'(t)$ are continuous and not simultaneously zero for $a < t < b$)

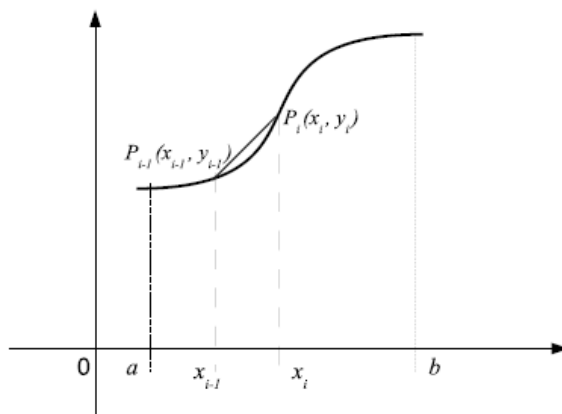
Let P be a partition of $[a, b]$ into n subintervals of equal length Δt .

$$a = t_0 < t_1 < \dots < t_n = b, \quad t_i = a + i\Delta t$$

Point $P_i(x(t_i), y(t_i))$ lies on C and the polygon with vertices P_0, P_1, \dots, P_n approximates C .

We define the length of C to be

$$L = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n |P_{i-1} P_i|$$



Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1} P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Since

$$x'(t_i) \approx \frac{\Delta x_i}{\Delta t}, \quad y'(t_i) \approx \frac{\Delta y_i}{\Delta t},$$

then

$$\Delta x_i = x'(t_i)\Delta t, \quad \Delta y_i = y'(t_i)\Delta t$$

Thus,

$$L = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2} \Delta t = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = L$$

$x = x(t)$
 $y = y(t)$
 $a \leq t \leq b$

If the curve C is given by the equation

$$y = y(x), \quad a \leq x \leq b, \quad \text{then} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

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If the curve C is given by the equation

$$x = x(y), \quad c \leq y \leq d, \quad \text{then} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2} dy$$

Example 1. Find the length of the curve

(a) $x'(t) = 3 - 3t^2$, $y'(t) = 6t$
 $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 2$

$$\begin{aligned} L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2} dt \\ &= \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt = \int_0^2 \sqrt{9 + 18t^2 + 9t^4} dt = \int_0^2 \sqrt{9(1 + 2t^2 + t^4)} dt \\ &= 3 \int_0^2 \sqrt{1 + 2t^2 + t^4} dt = 3 \int_0^2 \sqrt{(1+t^2)^2} dt = 3 \int_0^2 (1+t^2) dt \\ &= 3 \left(t + \frac{t^3}{3} \right)_0^2 = 3 \left(2 + \frac{8}{3} \right) = \boxed{14} \end{aligned}$$

$$y'(x) = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2} \left(x^2 - \frac{1}{x^2} \right)$$

(b) $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \leq x \leq 2$

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + [y'(x)]^2} dx = \int_1^2 \sqrt{1 + \left[\frac{1}{2} \left(x^2 - \frac{1}{x^2} \right) \right]^2} dx \\ &= \int_1^2 \sqrt{1 + \frac{1}{4} \left(x^4 - 2 \cancel{x^2} \frac{1}{x^2} + \frac{1}{x^4} \right)} dx \\ &= \int_1^2 \sqrt{1 + \frac{1}{4} x^4 - \frac{2}{4} + \frac{1}{4x^4}} dx = \int_1^2 \sqrt{\frac{1}{4} x^4 + \frac{1}{2} + \frac{1}{4x^4}} dx \\ &= \int_1^2 \sqrt{\frac{1}{4} \left(x^4 + 2 + \frac{1}{x^4} \right)} dx = \frac{1}{2} \int_1^2 \sqrt{\left(x^2 + \frac{1}{x^2} \right)^2} dx \\ &= \frac{1}{2} \int_1^2 \left(x^2 + \frac{1}{x^2} \right) dx = \frac{1}{2} \left(\frac{x^3}{3} - \frac{1}{x} \right)_1^2 \\ &= \boxed{\frac{1}{2} \left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \right)} \end{aligned}$$

$$x'(y) = \frac{3}{2} y^{1/2}$$

$$(c) x = y^{3/2}, 0 \leq y \leq 1$$

$$L = \int_0^1 \sqrt{1 + [x'(y)]^2} dy = \int_0^1 \sqrt{1 + \left(\frac{3}{2} y^{1/2}\right)^2} dy = \int_0^1 \sqrt{1 + \frac{9}{4} y} dy$$

$$= \frac{4}{9} \int_1^{13/4} \sqrt{u} du = \frac{4}{9} \left[\frac{u^{3/2}}{3/2} \right]_1^{13/4} = \frac{8}{27} \left(\left(\frac{13}{4}\right)^{3/2} - 1 \right)$$

$$= \boxed{\frac{8}{27} \left(\frac{13\sqrt{13}}{8} - 1 \right)}$$

$$\left. \begin{array}{l} u = 1 + \frac{9}{4} y \\ du = \frac{9}{4} dy \\ dy = \frac{4}{9} du \\ 0 \rightarrow 1 + \frac{9}{4}(0) = 1 \\ 1 \rightarrow 1 + \frac{9}{4}(1) = \frac{13}{4} \end{array} \right|$$