Section 9.4 **Area of a surface of revolution**

A surface of revolution is formed when a curve is rotated about a line.

Let's start with some simple surfaces.

The lateral surface area of a circular cylinder with base radius r and height h is

$$A = 2\pi rh$$

The lateral surface area of a circular cone with base radius r and slant height l is

$$A = \pi rl$$

The lateral surface area of a band (frustum of a cone) with slant height l, upper radius r_1 and and lower radius r_2 is

$$A = 2\pi rl, \text{ here } r = \frac{r_1 + r_2}{2}$$

Now we consider the surface which is obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$ about the x-axis, $f(x) > 0$ for all x in $[a, b]$ and $f'(x)$ is continuous.

We take a partition P of $[a, b]$ by points $a = x_0 < x_1 < ... < x_n = b$, and let $y_i = f(x_i)$, so that the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $|P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$, its surface area is

$$S_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

We know that

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x_i$$

where $x_i^* \in [x_{i-1}, x_i]$. Since Δx_i is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ since f is continuous.

$$S_i \approx 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x_i$$

Thus, the area of the complete surface is

$$S_X = 2\pi \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\Delta x_i =$$

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi \int_a^b f(x)\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$S_X = 2\pi \int_c^d y\sqrt{1+[g'(y)]^2}dy = 2\pi \int_c^d y\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$$

Let's a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

The area of the surface generated by rotating C about x-axis is

$$S_X = 2\pi \int_a^b y(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt$$

Example 1. Find the area of the surface obtained by rotating the curve about x-axis

(a) $y = \sqrt{x}$, $4 \le x \le 9$

(b) $y^2 = 4x + 4$, $0 \leq x \leq 8$

(c) $x(t) = a\cos^3 t$, $y(t) = a\sin^3 t$, $0 \leq t \leq \pi/2$, a is a constant.

For rotation about the y-axis, the surface area formulas are:

if the curve is given as $y = f(x)$, $a \leq x \leq b$, then the formula for the surface area is

$$S_Y = 2\pi \int_a^b x\sqrt{1 + \left[\frac{df}{dx}\right]^2}\,dx$$

if the curve is described as $x = g(y)$, $c \leq y \leq d$, then the formula for the surface area is

$$S_Y = 2\pi \int_c^d g(y)\sqrt{1 + \left[\frac{dg}{dy}\right]^2}\,dy$$

if the is defined by the equations $x = x(t)$,$y = y(t)$, $a \leq t \leq b$, then the area of the surface is

$$S_Y = \int_a^b x(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt$$

Example 2. Find the area of the surface obtained by rotating the curve about y-axis

(a) $x = \sqrt{2y - y^2}$, $0 \le y \le 1$

(b) $y = 1 - x^2$, $0 \le x \le 1$

(c) $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$