Section 9.4 **Area of a surface of revolution**

A surface of revolution is formed when a curve is rotated about a line.

Let's start with some simple surfaces.

The lateral surface area of a circular cylinder with base radius r and height h is

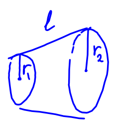
$$A = 2\pi rh$$

The lateral surface area of a circular cone with base radius r and slant height l is

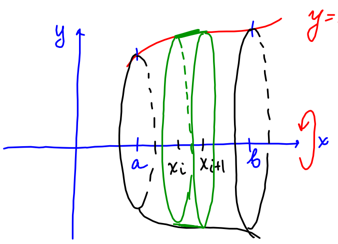
$$A = \pi rl$$

The lateral surface area of a band (frustum of a cone) with slant height l, upper radius r_1 and and lower radius r_2 is

$$A = 2\pi rl, \text{ here } r = \frac{r_1 + r_2}{2}$$



Now we consider the surface which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis, $f(x) > 0$ for all x in $[a, b]$ and $f'(x)$ is continuous.



We take a partition P of $[a, b]$ by points $a = x_0 < x_1 < ... < x_n = b$, and let $y_i = f(x_i)$, so that the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $|P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$, its surface area is

$$S_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

We know that

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x_i$$

where $x_i^* \in [x_{i-1}, x_i]$. Since Δx_i is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ since f is continuous.

$$S_i \approx 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x_i$$

Thus, the area of the complete surface is

$$S_X = 2\pi \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\Delta x_i =$$

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}dx = \boxed{2\pi \int_a^b f(x)\sqrt{1+\left[\frac{dy}{dx}\right]^2}dx = S.A.}$$

C: $y=f(x)$, $a \le x \le b$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$S_X = 2\pi \int_c^d y\sqrt{1+[g'(y)]^2}dy = \boxed{2\pi \int_c^d y\sqrt{1+\left[\frac{dx}{dy}\right]^2}dy = S.A.}$$

Let's a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

The area of the surface generated by rotating C about x-axis is

$$\boxed{S_X = 2\pi \int_a^b y(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}dt = S.A.}$$

Example 1. Find the area of the surface obtained by rotating the curve about x-axis

$y'(x) = \frac{1}{2}x^{-1/2}$

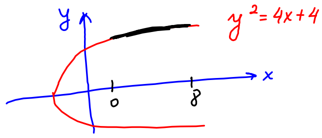
(a) $y = \sqrt{x}$, $4 \le x \le 9$

$$S.A. = 2\pi \int_4^9 y(x)\sqrt{1+[y'(x)]^2}\,dx = 2\pi \int_4^9 \sqrt{x}\sqrt{1+\left(\tfrac{1}{2}x^{-1/2}\right)^2}\,dx$$

$$= 2\pi \int_4^9 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi \int_4^9 \sqrt{x\left(1+\frac{1}{4x}\right)}\,dx = 2\pi \int_4^9 \sqrt{x+\frac{1}{4}}\,dx$$

$$= 2\pi \left.\frac{(x+1/4)^{3/2}}{3/2}\right]_4^9 = \frac{4\pi}{3}\left(\left(\frac{37}{4}\right)^{3/2} - \left(\frac{17}{4}\right)^{3/2}\right)$$

$$= \frac{4\pi}{3}\left(\frac{37\sqrt{37}}{8} - \frac{17\sqrt{17}}{8}\right) = \boxed{\frac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right)}$$



$y>0$

(b) $y^2 = 4x + 4$, $0 \le x \le 8$

$x = \frac{y^2-4}{4} = \frac{y^2}{4} - 1$, $x'(y) = \frac{2y}{4} = \frac{y}{2}$

$x=0 \Rightarrow y^2 = 4 \Rightarrow y = 2$

$x=8 \Rightarrow y^2 = 36 \Rightarrow y = 6$

$2 \le y \le 6$

$$S.A. = 2\pi \int_2^6 y\sqrt{1+[x'(y)]^2}\,dy = 2\pi \int_2^6 y\sqrt{1+\left(\frac{y}{2}\right)^2}\,dy = 2\pi\int_2^6 y\sqrt{1+\frac{y^2}{4}}\,dy$$

$$= 2\pi\int_2^6 y\sqrt{\frac{4+y^2}{4}}\,dy = \frac{2\pi}{2}\int_2^6 y\sqrt{4+y^2}\,dy$$

$u = 4+y^2$
$du = 2y\,dy \Rightarrow y\,dy = \frac{du}{2}$
$2 \to 4+2^2 = 8$
$6 \to 4+6^2 = 40$

$$= \frac{\pi}{2}\int_8^{40}\sqrt{u}\,du = \frac{\pi}{2}\left.\frac{u^{3/2}}{3/2}\right]_8^{40} = \frac{\pi}{3}\left(40^{3/2} - 8^{3/2}\right)$$

$$= \frac{\pi}{3}\left(40\underbrace{\sqrt{40}}_{(4)(10)} - 8\underbrace{\sqrt{8}}_{(2)(4)}\right) = \boxed{\frac{\pi}{3}\left(80\sqrt{10} - 16\sqrt{2}\right)}$$

$x'(t) = 3a\cos^2(t)(-\sin t)$ $\quad y'(t) = 3a\sin^2 t\cos t$

(c) $x(t) = a\cos^3 t$, $y(t) = a\sin^3 t$, $0 \le t \le \pi/2$, a is a constant.

$$S.A. = 2\pi\int_0^{\pi/2} y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$= 2\pi\int_0^{\pi/2} a\sin^3 t\sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\,dt$$

$$= 2\pi\int_0^{\pi/2} a\sin^3 t\sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$$

$$= 2\pi\int_0^{\pi/2} a\sin^3 t\sqrt{9a^2\cos^2 t\sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}\,dt$$

$$= 2\pi\int_0^{\pi/2} a\sin^3 t(3a\cos t\sin t)\,dt = 3a^2(2\pi)\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

$u = \sin t$
$du = \cos t\,dt$
$0 \to \sin 0 = 0$
$\frac{\pi}{2} \to \sin\frac{\pi}{2} = 1$

$$= 6\pi a^2\int_0^1 u^4\,du = 6\pi a^2\left.\frac{u^5}{5}\right]_0^1 = \boxed{\frac{6\pi a^2}{5}}$$

For rotation about the y-axis, the surface area formulas are:

if the curve is given as $y = f(x)$, $a \le x \le b$, then the formula for the surface area is

$$\boxed{S_Y = 2\pi \int_a^b x\sqrt{1 + \left[\frac{df}{dx}\right]^2}\, dx = S.A.}$$

if the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$\boxed{S_Y = 2\pi \int_c^d g(y)\sqrt{1 + \left[\frac{dg}{dy}\right]^2}\, dy = S.A.}$$

if the is defined by the equations $x = x(t)$,$y = y(t)$, $a \le t \le b$, then the area of the surface is

$$\boxed{S_Y = 2\pi\int_a^b x(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = S.A.}$$

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Example 2. Find the area of the surface obtained by rotating the curve about y-axis

(a) $x = \sqrt{2y - y^2}$, $0 \le y \le 1$, $x'(y) = \frac{1}{2}(2y-y^2)^{-1/2}(2-2y) = \frac{1-y}{\sqrt{2y-y^2}}$

$$S.A. = 2\pi\int_0^1 x(y)\sqrt{1+(x'(y))^2}\, dy$$

$$= 2\pi\int_0^1 \sqrt{2y-y^2}\sqrt{1+\left[\frac{1-y}{\sqrt{2y-y^2}}\right]^2}\, dy = 2\pi\int_0^1 \sqrt{2y-y^2}\sqrt{1+\frac{(1-y)^2}{2y-y^2}}\, dy$$

$$= 2\pi\int_0^1 \sqrt{2y-y^2}\sqrt{\frac{\cancel{2y}-\cancel{y^2}+(1-\cancel{2y}+\cancel{y^2})}{2y-y^2}}\, dy = 2\pi\int_0^1 \cancel{\sqrt{2y-y^2}}\frac{1}{\cancel{\sqrt{2y-y^2}}}\, dy$$

$$= 2\pi\int_0^1 dy = 2\pi y\Big]_0^1 = \boxed{2\pi}$$

$y'(x) = -2x$

(b) $y = 1 - x^2$, $0 \le x \le 1$

$$S.A. = 2\pi\int_0^1 x\sqrt{1+(y'(x))^2}\, dx = 2\pi\int_0^1 x\sqrt{1+(-2x)^2}\, dx = 2\pi\int_0^1 x\sqrt{1+4x^2}\, dx$$

$u = 1+4x^2$
$du = 8x\,dx$
$x\,dx = \frac{du}{8}$
$0 \to 1+4(0)^2 = 1$
$1 \to 1+4(1)^2 = 5$

$$= \frac{2\pi}{8}\int_1^5 \sqrt{u}\, du = \frac{\pi}{4}\frac{u^{3/2}}{3/2}\Big]_1^5 = \frac{\pi}{6}\left[5^{3/2} - 1^{3/2}\right]$$

$$= \boxed{\frac{\pi}{6}\left[5\sqrt{5}-1\right]}$$

$x'(t)=e^t-1$, $y'(t)=4e^{t/2}(\frac{1}{2})=2e^{t/2}$

(c) $x=e^t-t$, $y=4e^{t/2}$, $0\le t\le 1$

$$S.A.=2\pi\int_0^1 x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

$$=2\pi\int_0^1(e^t-t)\sqrt{\underbrace{(e^t-1)^2}_{e^{2t}-2e^t+1}+\underbrace{(2e^{t/2})^2}_{4e^t}}\,dt=2\pi\int_0^1(e^t-t)\sqrt{e^{2t}-2e^t+1+4e^t}\,dt$$

$$=2\pi\int_0^1(e^t-t)\sqrt{\underbrace{e^{2t}+2e^t+1}_{(e^t+1)^2}}\,dt=2\pi\int_0^1(e^t-t)\sqrt{(e^t+1)^2}\,dt$$

$$=2\pi\int_0^1(e^t-t)(e^t+1)\,dt=2\pi\int_0^1(e^{2t}+e^t-te^t-t)\,dt$$

$$=2\pi\left[\int_0^1 e^{2t}dt+\int_0^1 e^t dt-\int_0^1 te^t dt-\int_0^1 t\,dt\right]$$

$$\boxed{\int uv'dx=uv-\int vu'dx}$$

$$=2\pi\left(\frac{1}{2}e^{2t}\Big]_0^1+e^t\Big]_0^1-\frac{t^2}{2}\Big]_0^1-\int_0^1 te^t dt\right)$$

$u=t$ $\quad u'=1$

$v'=e^t$ $\quad v=e^t$

$$=2\pi\left(\frac{1}{2}(e^2-1)+(e-1)-\frac{1}{2}-\left\{te^t\Big]_0^1-\int_0^1 e^t dt\right\}\right)$$

$$=2\pi\left(\frac{1}{2}(e^2-1)+(e-1)-\frac{1}{2}-\left\{e-e^t\Big]_0^1\right\}\right)$$

$$=2\pi\left(\frac{1}{2}e^2+\cancel{e}-2-\cancel{e}+e-1\right)$$

$$=\boxed{2\pi\left(\frac{1}{2}e^2+e-3\right)}$$