10.7: Taylor and Maclaurin Series

- **The Taylor series for $f(x)$ about $x = a$:**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
- **The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
- *Known Mclaurin series and their intervals of convergence you must have memorized:*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad (1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1, 1]$$

Examples.

1. Given that function f has power series expansion (i.e. Taylor series) centered at $a = \pi$. Find this expansion and its radius of convergence if it is given that
$$f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}.$$
2. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$.
3. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?
4. Find Taylor series for $f(x) = \dfrac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?
5. Find Maclaurin series for the following functions:
   (a) $f(x) = x^3 \sin x^5$

(b) $f(x) = \sin^2 x$

(c) $x + 3x^2 + xe^{-x}$

6. Express $\displaystyle\int \frac{\sin(3x)}{x}\,\mathrm{d}x$ as an infinite series.

7. Find the sum of the series:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{n!}$

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

8. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x$ with error less than 10^{-3}.

10.9: Applications of Taylor Polynomials

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x)\\ N-\text{th degree}\\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x)\\ \textbf{Remainder}}}$$

Examples.

9. Find the fourth-degree Taylor polynomial of $f(x) = \dfrac{1}{2+6x}$ centered at $a = 0$.

10. Find the third-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$.

11. Find the second degree Taylor Polynomial for $f(x) = \ln x$ at $a = 3$.

11.1: Three-dimensional Coordinate System

- The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is
$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
- Equation of a sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ (completing the square)

Examples.

12. Graph the following regions:
    (a) $x = 5$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$; **(b)** $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.

13. Given the sphere $(x-1)^2 + (y+4)^2 + (z-2)^2 = 16$.

    (a) What is the intersection of the sphere with the yz-plane.

    (b) Find the distance from the point $(1, -2, 3)$ to the center of the sphere.

14. What is the intersection of the surface $x^2 + y^2 = 49$ with the xy-plane.

15. Determine the radius and the center of the sphere given by the equation
$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$