Math 152/172 **WEEK in REVIEW 10** Spring 2016

Sections 10.7, 10.9, 11.1

10.7: Taylor and Maclaurin Series

- The Taylor series for $f(x)$ about $x = a$:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =}$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + \dots \quad (-1, 1)$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (-\infty, \infty)$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty)$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty)$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad [-1, 1]$$

$$\arctan x = \int \frac{dx}{1+x^2} = \int \left(\sum_{n=0} (-1)^n x^{2n}\right) dx = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\arctan 0 = 0 = C$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Examples.

1. Given that function f has power series expansion (i.e. Taylor series) centered at $a = \pi$. Find this expansion and its radius of convergence if it is given that

$$f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= \sum_{n=0}^{\infty} \frac{\overbrace{\frac{(-1)^n n!}{4^{2n+1}(2n+1)!}}^{= f^{(n)}(\pi)}}{n!}(x-\pi)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{4^{2n+1}(2n+1)!\,\cancel{n!}}(x-\pi)^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n+1}(2n+1)!}(x-\pi)^n}$$

Radius of convergence (do the Ratio Test), $a_n = \frac{(-1)^n (x-\pi)^n}{4^{2n+1}(2n+1)!}$, $a_{n+1} = \frac{(-1)^{n+1}(x-\pi)^{n+1}}{4^{2(n+1)+1}(2(n+1)+1)!}$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-\pi)^{n+1}}{4^{2(n+1)+1}(2(n+1)+1)!}}{\frac{(-1)^n (x-\pi)^n}{4^{2n+1}(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cancel{4^{2n+1}}(2n+1)!(x-\pi)}{\cancel{(-1)^n}\,4^{\cancel{2n+3}\,2}(2n+3)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\cancel{(2n+1)(2n)(2n-1)\dots(1)}(x-\pi)}{4^2(2n+3)(2n+2)\cancel{(2n+1)\dots(2)(1)}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-\pi)}{4^2(2n+3)(2n+2)}\right| = 0 < 1 \text{ for all } x$$

series converges for all $-\infty < x < \infty$, $\boxed{R = \infty}$

2. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \Rightarrow f^{(n)}(0) = 1$$

where $f(x) = e^{x^2}$

$$\boxed{f^{(20)}(0) = 1}$$

3. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1/3)}{n!} \left(x - \frac{1}{3}\right)^n$$

$f(x) = e^{3x} = 3^0 \cdot e^{3x}$

$f'(x) = 3e^{3x} = 3^1 \cdot e^{3x}$

$f''(x) = 3(3)e^{3x} = 3^2 e^{3x}$

$f'''(x) = 3^2(3)e^{3x} = 3^3 e^{3x}$

$\Rightarrow f^{(n)}(x) = 3^n e^{3x}$

$f^{(n)}\left(\frac{1}{3}\right) = 3^n e^{3\left(\frac{1}{3}\right)} = 3^n e$

$$\boxed{e^{3x} = \sum_{n=0}^{\infty} \frac{3^n e}{n!} \left(x - \frac{1}{3}\right)^n}$$

Radius of convergence: $a_n = \dfrac{3^n e}{n!} \left(x - \frac{1}{3}\right)^n, \quad a_{n+1} = \dfrac{3^{n+1} e}{(n+1)!} \left(x - \frac{1}{3}\right)^{n+1}$

$(n+1)! = (n+1)\, n!$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1} e}{(n+1)!} \left(x - \frac{1}{3}\right)^{n+1}}{\frac{3^n e}{n!} \left(x - \frac{1}{3}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{3^{\cancel{n+1}} \cancel{e} \left(x - \frac{1}{3}\right)^{\cancel{n+1}} n!}{\cancel{3^n} \cancel{e} \cancel{\left(x - \frac{1}{3}\right)^n} (n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(x - \frac{1}{3}\right) \cancel{n!}}{(n+1) \cancel{n!}} \right| = \lim_{n\to\infty} \left| \frac{\left(x - \frac{1}{3}\right)}{n+1} \right| = 0 \text{ for all } x.$$

$$\boxed{R = \infty}$$

4. Find Taylor series for $f(x) = \dfrac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?

$$f(x) = x^{-1}$$

$$f'(x) = (-1)x^{-2}$$

$$f''(x) = (-1)(-2)x^{-3} = (-1)^{\boxed{2}}(\boxed{2}!)x^{-\boxed{3}} \quad n=2$$

$$f'''(x) = (-1)^2(-3)(2!)x^{-4} = (-1)^{\boxed{3}}\boxed{3}!\,x^{-\boxed{4}} \quad n=3$$

$$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}$$

$$f^{(n)}(5) = \frac{(-1)^n n!}{5^{n+1}}$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n$$

$$= \sum_{n=0}^{\infty} \frac{\frac{(-1)^n \cancel{n!}}{5^{n+1}}}{\cancel{n!}} (x-5)^n \qquad = f^{(n)}(5)$$

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}(x-5)^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{5^{n+2}}(x-5)^{n+1}}{\frac{(-1)^n}{5^{n+1}}(x-5)^n}\right| = \frac{|x-5|}{5} < 1 \Rightarrow |x-5| < 5 \qquad \boxed{R=5}$$

5. Find Maclaurin series for the following functions:

(a) $f(x) = x^3 \sin(x^5)$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\sin(x^5) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^5)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}}{(2n+1)!}$

$x^3 \sin(x^5) = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}\, x^3}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+8}}{(2n+1)!}}$

(b) $f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$

$\frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!} = \frac{1}{2} - \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}\right] = \frac{1}{2} - \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$

$= \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right) \frac{2^{2n} x^{2n} (-1)^n}{(2n)!}$

$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}}$

(c) $x + 3x^2 + xe^{-x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$xe^{-x} = x\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$

$x + 3x^2 + xe^{-x} = \boxed{x + 3x^2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}} = x + 3x^2 + \left[x - x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n x^{n+1}}{n!}\right]$

$= \boxed{2x + 2x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n x^{n+1}}{n!}}$

6. Express $\int \frac{\sin(3x)}{x}\, dx$ as an infinite series.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(3x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin(3x)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin(3x)}{x}\, dx = \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n}}{(2n+1)!} \right) dx = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} \left(\int x^{2n}\, dx \right)$$

$$= \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)(2n+1)!}}$$

7. Find the sum of the series:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

(b) $\sum_{n=0}^{\infty} \frac{7^n}{n!}$ $= e^7$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$ $= \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \arctan(x^3)$

8. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\,dx$ with error less than 10^{-3}.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^{0.5} x^{2n+2}\,dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x^{2n+3}}{2n+3} \right)_0^{0.5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+3)\,2^{2n+3}}$$

$$= \underbrace{\frac{1}{3 \cdot 2^3}}_{n=0} - \underbrace{\frac{1}{5 \cdot 2^5}}_{n=1} + \underbrace{\cancel{\frac{1}{2(7)2^7}}}_{n=2} - \underbrace{\cancel{\frac{1}{3!(9)2^9}}}_{n=3} + \cdots$$

$$\frac{1}{160} \qquad \frac{1}{1792} < 10^{-3}$$

$$\approx \boxed{\frac{1}{24} - \frac{1}{160}}$$

$T_N(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^N(a)}{N!}(x-a)^N$

10.9: Applications of Taylor Polynomials

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

Examples.

9. Find the fourth-degree Taylor polynomial of $f(x) = \dfrac{1}{2+6x}$ centered at $a = 0$.

$f(x) = \frac{1}{2+6x} = \frac{1}{2(1+3x)} = \frac{1}{2}\sum_{n=0}^{\infty}(-3x)^n = \frac{1}{2}\left[1 - 3x + 9x^2 - 27x^3 + 81x^4 + \dots\right]$

$\boxed{T_4(x) = \frac{1}{2}\left(1 - 3x + 9x^2 - 27x^3 + 81x^4\right)}$

10. Find the third-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$.

$f(x) = x^{1/3}$

$f(x) = x^{1/3}$	$f(1) = 1$
$f'(x) = \frac{1}{3}x^{-2/3}$	$f'(1) = \frac{1}{3}$
$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3}$	$f''(1) = -\frac{2}{9}$
$f'''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^{-8/3}$	$f'''(1) = \frac{10}{27}$

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$= 1 + \frac{1}{3}(x-1) - \frac{2}{(9)(2)}(x-1)^2 + \frac{10}{27(6)}(x-1)^3$$

$$= \boxed{1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3}$$

11. Find the second degree Taylor Polynomial for $f(x) = \ln x$ at $a = 3$.

$$T_2(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2$$

$f(x) = \ln x$	$f(3) = \ln 3$
$f'(x) = \frac{1}{x}$	$f'(3) = \frac{1}{3}$
$f''(x) = -\frac{1}{x^2}$	$f''(3) = -\frac{1}{9}$

$$\boxed{T_2(x) = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2}$$

11.1: Three-dimensional Coordinate System

- The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- Equation of a sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ (completing the square)

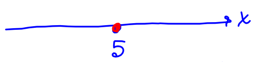
center @ (a,b,c) of radius r

Examples.

12. Graph the following regions:

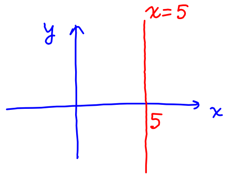
(a) $x = 5$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$;

$\mathbb{R}$:



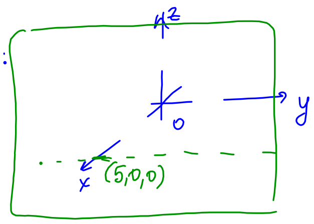
$x=5$ (point)

$\mathbb{R}^2$:



$x=5$ line parallel to y-axis through $(5,0)$

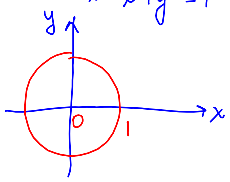
$\mathbb{R}^3$:



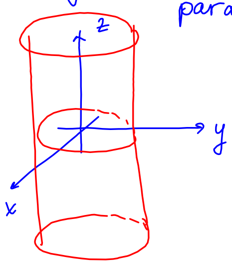
$x=5$ -plane parallel to the (yz)-plane through $(5,0,0)$

(b) $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.

$\mathbb{R}^2$: $x^2+y^2=1$ circle centered at $(0,0)$ of radius 1



$\mathbb{R}^3$: $x^2+y^2=1$ circular cylinder parallel to z-axis



13. Given the sphere $(x-1)^2+(y+4)^2+(z-2)^2=16$.

(a) What is the intersection of the sphere with the yz-plane.

$x=0$ - equation of yz-plane.

$$\begin{cases} x=0 \\ (x-1)^2+(y+4)^2+(z-2)^2=16 \end{cases}$$

$$(-1)^2+(y+4)^2+(z-2)^2=16$$

$$\boxed{(y+4)^2+(z-2)^2=15}$$ circle.

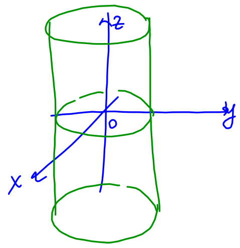
(b) Find the distance from the point $(1,-2,3)$ to the center of the sphere.

center of the sphere $(1,-4,2)$

distance between $(1,-2,3)$ and $(1,-4,2)$

$$d=\sqrt{(1-1)^2+(-4-(-2))^2+(2-3)^2}=\sqrt{2^2+1^2}=\sqrt{5}$$

14. What is the intersection of the surface $\underbrace{x^2 + y^2 = 49}_{\text{cylinder}}$ with the $\underbrace{xy}_{z=0}$-plane.



circle $x^2+y^2=49$ centered at the origin of radius 7.

15. Determine the radius and the center of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$

Complete squares.

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$$x^2 + (y^2 + \overbrace{2y}^{2(1)y}) + (z^2 + \overbrace{z}^{2(\frac{1}{2})z}) - 1 = 0$$

$$x^2 + (y^2 + 2y + 1) - 1 + (z^2 + z + (\tfrac{1}{2})^2) - (\tfrac{1}{2})^2 - 1 = 0$$

$$x^2 + (y+1)^2 + (z + \tfrac{1}{2})^2 - \frac{9}{4} = 0$$

$$x^2 + (y+1)^2 + (z + \tfrac{1}{2})^2 = \frac{9}{4}$$

$$\boxed{\text{center } (0, -1, -\tfrac{1}{2}), \quad \text{radius} = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}}$$