10.7: Taylor and Maclaurin Series

- **The Taylor series for $f(x)$ about $x=a$:**
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=$$
$$=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\dots$$
- **The Maclaurin series is the Taylor series about $x=0$ (i.e. a=0):**
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3+\dots \qquad (1,1)$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots \qquad (-\infty,\infty)$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots \qquad (-\infty,\infty)$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots \qquad (-\infty,\infty)$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad [-1,1]$$

Examples.

1. Given that function f has power series expansion (i.e. Taylor series) centered at $a=\pi$. Find this expansion and its radius of convergence if it is given that
$$f^{(n)}(\pi)=\frac{(-1)^n n!}{4^{2n+1}(2n+1)!}.$$

2. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$.

3. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?

4. Find Taylor series for $f(x) = \dfrac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?

5. Find Maclaurin series for the following functions:

   (a) $f(x) = x^3 \sin x^5$

   (b) $f(x) = \sin^2 x$

   (c) $x + 3x^2 + xe^{-x}$

6. Express $\displaystyle\int \frac{\sin(3x)}{x}\, \mathrm{d}x$ as an infinite series.

7. Find the sum of the series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

8. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x$ with error less than 10^{-3}.

10.9: Applications of Taylor Polynomials

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

Examples.

9. Find the fourth-degree Taylor polynomial of $f(x) = \dfrac{1}{2+6x}$ centered at $a = 0$.

10. Find the third-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$.

11. Find the second degree Taylor Polynomial for $f(x) = \ln x$ at $a = 3$.

11.1: Three-dimensional Coordinate System

- The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- Equation of a sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ (completing the square)

Examples.

12. Graph the following regions:
(a) $x = 5$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$;

(b) $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.

13. Given the sphere $(x - 1)^2 + (y + 4)^2 + (z - 2)^2 = 16$.

(a) What is the intersection of the sphere with the yz-plane.

(b) Find the distance from the point $(1, -2, 3)$ to the center of the sphere.

14. What is the intersection of the surface $x^2 + y^2 = 49$ with the xy-plane.

15. Determine the radius and the center of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$