Math 152/172 **WEEK in REVIEW 11** Spring 2016

Review for Test 3, covering 10.3–10.7, 10.9, 11.1

1. Which of the following series converges absolutely?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}}$

$-1 < \sin x < 1$

$-1 < \cos x < 1$

$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

Comparison Test.

$-1 < \sin(\pi^3 n^2) < 1$

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}} \right|$, $\quad \dfrac{0}{n^2\sqrt{n}} \le \dfrac{|\sin(\pi^3 n^2)|}{n^{5/2}} \le \dfrac{1}{n^{5/2}}$

$$\frac{|\sin(\pi^3 n^2)|}{n^{5/2}} \le \frac{1}{n^{5/2}}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ $(p = \frac{5}{2} > 1)$ is convergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(\pi^3 n^2)|}{n^2\sqrt{n}}$ is convergent by Comparison Test.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}}$ is $\boxed{\text{absolutely convergent}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ - alternating series, $b_n = \frac{1}{\sqrt[4]{n}}$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[4]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ $(p = \frac{1}{4} < 1)$ divergent

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$

>0 for $n \geq 1$

- $(n^{-1/4})' = -\frac{1}{4} n^{-5/4} = - \frac{1}{4} \frac{1}{\sqrt[4]{n^5}} < 0$ on $[1, \infty)$

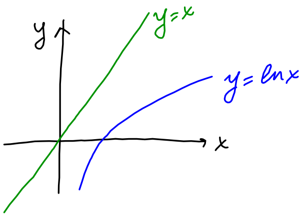
$b_n = n^{-1/4}$ is decreasing.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ is convergent by Alternating series test.

convergent but not absolutely convergent = $\boxed{\text{conditionally convergent}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right|$

$\ln x < x$



$\ln n < n$

$\dfrac{1}{\ln n} > \dfrac{1}{n}$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ harmonic series divergent

$\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right|$ is divergent by Comparison Test.

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is alternating for $b_n = \dfrac{1}{\ln n}$

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} \underbrace{=}_{\lim_{n\to\infty} \ln n = \infty} 0$
- $\ln n$ is increasing for all n, thus $\dfrac{1}{\ln n}$ is decreasing

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by Alternating Series Test.

$\boxed{\text{Conditionally convergent}}$

(d) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

Ratio Test. for $a_n = \frac{n^n}{(n!)^2}$, $a_{n+1} = \frac{(n+1)^{n+1}}{[(n+1)!]^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{[(n+1)!]^2}}{\frac{n^n}{(n!)^2}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{[(n+1)!]^2}\cdot\frac{(n!)^2}{n^n}\right|$$

$(n+1)! = (n+1)\,n!$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}\cancel{(n!)^2}}{(n+1)^2\cancel{(n!)^2}\,n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{\cancel{n+1}}}{(n+1)^{\cancel{2}}\,n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^n}{(n+1)\,n^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\left(\cancel{\frac{n+1}{n}}\right)^n\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

$\boxed{\text{convergent absolutely}}$ by Ratio Test

(e) $\sum_{n=1}^{\infty} \frac{5^n}{\ln(n+1)}$

$a_n = \frac{5^n}{\ln(n+1)}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{\ln(n+1)} \overset{\text{L'Hospital's}}{=} \lim_{n\to\infty} \frac{5^n \ln 5}{\frac{1}{n+1}}$

$= \lim_{n\to\infty} (n+1)5^n \ln 5 = \infty \neq 0$

$\boxed{\text{Divergent}}$ by Divergence Test

(f) $\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$

Cannot do Comparison Test, do Limit Comparison Test instead

$a_n = \frac{n^2+4}{n^{11}+n^7+n+1}$ compare with $b_n = \frac{n^2}{n^{11}} = \frac{1}{n^9}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+4}{n^{11}+n^7+n+1}}{\frac{1}{n^9}} = \lim_{n\to\infty} \frac{n^2+4}{n^{11}+n^7+n+1} \cdot n^9$$

$$= \lim_{n\to\infty} \frac{n^9 n^2 \frac{n^2+4}{n^2}}{n^{11} \frac{n^{11}+n^7+n+1}{n^{11}}} = \lim_{n\to\infty} \frac{\cancel{n^{11}}\left(1+\cancel{\frac{4}{n^2}}^{\,0}\right)}{\cancel{n^{11}}\left(1+\cancel{\frac{1}{n^{11-7}}}^{\,0}+\cancel{\frac{1}{n^{11-1}}}^{\,0}+\cancel{\frac{1}{n^{11}}}^{\,0}\right)} = 1$$

$\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^9}$ either both converge or both diverge.

$\sum_{n=1}^{\infty} \frac{1}{n^9}$ $(p=9>1)$ is convergent, so is $\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$ by Limit Comparison Test

for all n $a_n>0$, thus $\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$ is $\boxed{\text{absolutely convergent}}$

2. Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-4)^n$ has the radius of convergence 4. Consider the following pair of series:

conv. $(x=7<8)$

$$(I)\ \sum_{n=1}^{\infty} c_n 5^n \qquad (II)\ \sum_{n=1}^{\infty} c_n 3^n.$$

Interval of convergence

$|x-4|<4$

$-4<x-4<4$

$\boxed{0<x<8}$

div. 0 conv. 8 diverg.

Which of the following statements is true?

$x-4=5 \Rightarrow x=9$

$x-4=3 \Rightarrow x=7$

(a) (I) is convergent, (II)is divergent

(b) Neither series is convergent

(c) Bith series are convergent

(d) (I) is divergent, (II) is convergent

(e) no conclusion can be drawn about either series.

$\sum_{n=1}^{\infty} c_n 5^n$ $x=9>8$ div.

3. Show that the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges. Then find an upper bound on the error in using s_{10} to approximate the series. (Note that $\ln 2 > 1/2$.)

Do the Integral Test.

$f(x)=\frac{\ln x}{x^2}$

Show that $f(x)=\frac{\ln x}{x^2}$ is positive, continuous, decreasing on $[2,\infty)$

- $\ln x > 0$ on $[2,\infty)$
  $x^2>0$ on $[2,\infty)$ $\Rightarrow$ $f(x)>0$ on $[2,\infty)$

- $\frac{\ln x}{x^2}$ has discontinuity at $x=0 \Rightarrow$ continuous on $[2,\infty)$

- $f'(x)=\frac{\frac{1}{x}x^2-\ln x\,(2x)}{x^4} = \frac{x-2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3} < 0$

  since $\ln x > \frac{1}{2}$ on $[2,\infty)$ and $x \geq 2$

Thus, $f(x)=\frac{\ln x}{x^2}$ is decreasing on $[2,\infty)$

$$\int_2^\infty \frac{\ln x}{x^2}dx = \lim_{t\to\infty}\int_2^t \frac{\ln x}{x^2}dx \quad \left| \begin{array}{ll} u=\ln x & v'=x^{-2} \\ u'=\frac{1}{x} & v=-x^{-1} \end{array} \right|$$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{x}\Big|_2^t - \int_2^t (-x^{-1})\frac{1}{x}dx\right] = \lim_{t\to\infty}\left(-\frac{\ln t}{t} + \frac{\ln 2}{2} + \int_2^t \frac{1}{x^2}dx\right)$$

$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{x}\Big]_2^t\right) = \lim_{t\to\infty}\left(-\frac{\ln t}{t} + \frac{\ln 2}{2} - \cancel{\frac{1}{t}}^{\,0} + \frac{1}{2}\right)$$

$$= \frac{\ln 2}{2} + \frac{1}{2} + \lim_{t\to\infty}\left(-\frac{\ln t}{t}\right)$$

$$= \frac{\ln 2}{2} + \frac{1}{2} - \lim_{t\to\infty}\frac{\frac{1}{t}}{1} = \frac{\ln 2}{2} + \frac{1}{2} - \cancel{\lim_{t\to\infty}\frac{1}{t}}^{\,0} = \frac{\ln 2}{2} + \frac{1}{2} \text{ — convergent}$$

$\boxed{\sum_{n=2}^\infty \frac{\ln n}{n^2} \text{ converges}}$ by Integral Test.

Find an upper bound for R_{10}.

$$\sum_{n=2}^\infty \frac{\ln n}{n^2} \approx S_{10}$$

By Integral Test Remainder Estimate

$$\int_{11}^\infty f(x)dx \le \boxed{R_{10} \le \int_{10}^\infty f(x)dx}$$

$$\int_{10}^\infty \frac{\ln x}{x^2}dx = \frac{\ln 10}{10} + \frac{1}{10}$$

$$\boxed{R_{10} \le \frac{\ln 10}{10} + \frac{1}{10}}$$

Maclauren series

4. If we represent $\dfrac{x^2}{4+9x^2}$ as a power series centered at $a = 0$, what is the associated radius of convergence?

$$\frac{x^2}{4+9x^2} = \frac{x^2}{4} \cdot \frac{1}{1+\frac{9x^2}{4}} = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{9x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{2n}}{4^n}\frac{x^2}{4}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{2n+2}}{4^{n+1}}$$

$\sum_{n=0}^{\infty}\left(-\frac{9x^2}{4}\right)^n$ is the power series with $r = -\frac{9x^2}{4}$. Converges if $|r|<1$

$$\left|-\frac{9x^2}{4}\right| < 1$$

$$-1 < \frac{9x^2}{4} < 1 \quad \text{solve for } x.$$

$$-\frac{4}{9} < x^2 < \frac{4}{9}$$

$$\sqrt{|x^2|} < \sqrt{\frac{4}{9}} \Rightarrow |x| < \frac{2}{3} \Rightarrow \boxed{R = \frac{2}{3}}$$

5. Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n(3x-1)^n}{n}$.

Ratio Test, $a_n = \frac{(-2)^n(3x-1)^n}{n}$, $a_{n+1} = \frac{(-2)^{n+1}(3x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}(3x-1)^{n+1}}{n+1}}{\frac{(-2)^n(3x-1)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(3x-1)^{n+1}\,n}{(n+1)\cancel{(-2)^n}\cancel{(3x-1)^n}}\right|$$

$$= \lim_{n\to\infty}\left|-2(3x-1)\frac{n}{n+1}\right| = 2|3x-1| < 1$$

$$\frac{|3x-1|}{3} < \frac{\frac{1}{2}}{3} \Rightarrow |x-\tfrac{1}{3}| < \tfrac{1}{6} \Rightarrow \boxed{R = \tfrac{1}{6}}$$

Interval of convergence:

$$|x-\tfrac{1}{3}| < \tfrac{1}{6}$$
$$-\tfrac{1}{6} < x-\tfrac{1}{3} < \tfrac{1}{6}$$
$$-\tfrac{1}{6}+\tfrac{1}{3} < x < \tfrac{1}{6}+\tfrac{1}{3}$$
$$\tfrac{1}{6} < x < \tfrac{1}{2}$$

Test endpoints:

- $x = \frac{1}{6} \Rightarrow \sum_{n=1}^{\infty}\frac{(-2)^n(3\frac{1}{6}-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-2)^n(-\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ (harmonic series) divergent
- $x = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-2)^n(3\frac{1}{2}-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-2)^n(\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ – alternating series for $b_n = \frac{1}{n}$

$\left.\begin{array}{l}\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0 \\ \frac{1}{n} \text{ is decreasing}\end{array}\right\} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by alternating series Test

interval of convergence is $\boxed{(\tfrac{1}{6}, \tfrac{1}{2}]}$

16. Which of the following statements is TRUE?

~~(a)~~ If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

$\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent absolutely it is convergent.

~~(c)~~ If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(by alternating series Test $\lim_{n\to\infty} a_n = 0$ AND a_n should be decreasing)

~~(d)~~ If $a_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2} > 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

$\frac{e}{2} > 1$

7. Find a Maclaurin series representation for $\frac{e^x - 1 - x}{x^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$\frac{e^x - 1 - x}{x^2} = \frac{\left(1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}\right) - 1 - x}{x^2} = \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=2}^{\infty} \frac{x^{n-2}}{n!}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}}$$

8. (a) Find a Maclaurin series representation for $f(x) = \sin\left(\frac{x^2}{4}\right)$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin\left(\frac{x^2}{4}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{4}\right)^{2n+1} \frac{1}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{4^{2n+1}(2n+1)!}} \quad \text{on } (-\infty, \infty)$$

(b) Write $\int_0^1 \sin\left(\frac{x^2}{4}\right) dx$ as an infinite series.

$$\int_0^1 \sin\left(\frac{x^2}{4}\right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n+1}(2n+1)!}\left(\int_0^1 x^{4n+2}\, dx\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n+1}(2n+1)!} \left.\frac{x^{4n+3}}{4n+3}\right]_0^1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n+1}(2n+1)!\,(4n+3)}}$$

9. Let $f(x) = e^{5-x}$. Give the fourth degree Taylor polynomial for $f(x)$ centered around $a = 5$.

$$T_4(x) = f(5) + f'(5)(x-5) + \frac{f''(5)}{2}(x-5)^2 + \frac{f'''(5)}{2\cdot 3}(x-5)^3 + \frac{f^{iv}(5)}{2\cdot 3\cdot 4}(x-5)^4$$

$f(x) = e^{5-x}$	$f(5) = 1$
$f'(x) = -e^{5-x}$	$f'(5) = -1$
$f''(x) = e^{5-x}$	$f''(5) = 1$
$f'''(x) = -e^{5-x}$	$f'''(5) = -1$
$f^{iv}(x) = e^{5-x}$	$f^{iv}(5) = 1$

$$\boxed{T_4(x) = 1 + (-1)(x-5) + \frac{1}{2}(x-5)^2 - \frac{1}{6}(x-5)^3 + \frac{1}{24}(x-5)^4}$$

10 Find a Maclaurin series of $f(x) = \ln(2-x)$ and the associated radius of convergence.

$$-\int \frac{1}{2-x}\,dx = \ln(2-x) + C$$

$$\frac{1}{2-x} = \frac{1}{2}\,\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}} \Rightarrow \text{geometric series } r = \frac{x}{2}$$

$$\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \quad \boxed{R=2}$$

$$\ln(2-x) = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\left(\int x^n\,dx\right)$$

$$\ln(2-x) = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\,\frac{x^{n+1}}{n+1} + C$$

Plug $x = 0 \Rightarrow \ln 2 = C$

$$\boxed{\ln(2-x) = \ln 2 - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)2^{n+1}}} \quad \boxed{R=2}$$

18. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$ converges to s. Use the Alternating Series Theorem to estimate $|s - s_6|$.

$|s-s_6| = |R_6| \le b_7$, where $b_n = \frac{1}{(n^2)(3^n)}$

$$\boxed{|R_6| \le \frac{1}{7^2(3^7)} = \frac{1}{49(3^7)}}$$

12. Determine the radius and the center of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$

$x^2 + (y^2 + 2y) + (z^2 + z) - 1 = 0$

complete squares

$x^2 + (y^2 + 2y + 1) - 1 + \left(z^2 + 2\frac{1}{2}z + \left(\frac{1}{2}\right)^2\right) - \frac{1}{4} - 1 = 0$

$x^2 + (y+1)^2 + (z + \frac{1}{2})^2 = 2 + \frac{1}{4} = \frac{9}{4}$

center $(0, -1, -\frac{1}{2})$, $R = \frac{3}{2}$

13. Find the Taylor series for $f(x) = xe^{2x}$ at $x = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$f(x) = xe^{2x}$$

$$f'(x) = e^{2x} + 2xe^{2x}$$

$$f''(x) = 2e^{2x} + 2e^{2x} + 2x(2)e^{2x}$$

$$= \underbrace{4}_{2\cdot 2}e^{2x} + \underbrace{4}_{2^2}xe^{2x}$$

$$f'''(x) = 8e^{2x} + 4e^{2x} + 4x(2)e^{2x}$$

$$= \underbrace{12}_{3\cdot 4 = 3\cdot 2^2}e^{2x} + \overbrace{8}^{2^3}xe^{2x}$$

$$f^{iv}(x) = 24e^{2x} + 8e^{2x} + 8x(2)e^{2x}$$

$$= \underbrace{32}_{\substack{4\cdot 8 \\ 4\cdot 2^3}}e^{2x} + \underbrace{16}_{2^4}xe^{2x}$$

$$f^{(n)}(x) = n2^{n-1}e^{2x} + 2^n xe^{2x}$$

$$f^{(n)}(2) = n2^{n-1}e^4 + 2^n(2)e^4$$

$$= n2^{n-1}e^4 + 2^{n+1}e^4$$

$$\boxed{xe^{2x} = \sum_{n=0}^{\infty} \frac{n2^{n-1}e^4 + 2^{n+1}e^4}{n!}(x-2)^n}$$