$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b}$, where $F(x) = f(x)$

Math 152/172 WEEK in REVIEW 1 Spring 2016

1. Evaluate the definite integral.

(a) $\displaystyle\int_1^2 \frac{x^2+1}{\sqrt{x}}dx = \int_1^2 (x^2+1)\,x^{-1/2}dx = \int_1^2 \left(x^{2-1/2} + x^{-1/2}\right)dx$

$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$

$$= \int_1^2 x^{3/2}dx + \int_1^2 x^{-1/2}dx$$

$$= \frac{x^{3/2+1}}{3/2+1}\Big]_1^2 + \frac{x^{-1/2+1}}{-1/2+1}\Big]_1^2$$

$$= \frac{x^{5/2}}{5/2}\Big]_1^2 + \frac{x^{1/2}}{1/2}\Big]_1^2 = \frac{2}{5}x^{5/2}\Big]_1^2 + 2x^{1/2}\Big]_1^2 = \boxed{\frac{2}{5}\left(2^{5/2}-1\right) + 2\left(2^{1/2}-1\right)}$$

$2^{5/2} = 4(2)^{1/2}$

(b) $\displaystyle\int_1^8 \sqrt[3]{x}(x^2 - \sqrt{x})dx = \int_1^8 x^{1/3}(x^2 - \sqrt{x})dx = \int_1^8 (x^{1/3+2} - x^{1/3}\cdot x^{1/2})dx$

$$= \int_1^8 (x^{7/3} - x^{5/6})dx = \left[\frac{x^{7/3+1}}{7/3+1} - \frac{x^{5/6+1}}{5/6+1}\right]_1^8$$

$$= \left[\frac{x^{10/3}}{10/3} - \frac{x^{11/6}}{11/6}\right]_1^8 = \left[\frac{3}{10}x^{10/3} - \frac{6}{11}x^{11/6}\right]_1^8$$

$$= \frac{3}{10}8^{10/3} - \frac{6}{11}8^{11/6} - \frac{3}{10} + \frac{6}{11}$$

$$= \boxed{\frac{3}{10}(1024) - \frac{6}{11}8(8^{5/6}) - \frac{3}{10} + \frac{6}{11}}$$

(c) $\displaystyle\int_{-1}^2 |x - x^2|dx$

$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$

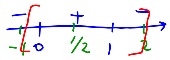
$x - x^2 > 0$

$x(1-x) > 0$

$2(1-2) = -2 < 0$

$\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4} > 0$

$-1(1-(-1)) = -2 < 0$



$$|x - x^2| = \begin{cases} x - x^2, & \text{if } x - x^2 > 0 \\ -(x-x^2), & \text{if } x - x^2 < 0 \end{cases}$$

$$= \begin{cases} x - x^2, & \text{if } 0 < x < 1 \\ -(x-x^2), & \text{if } x \le 0 \text{ or } x \ge 1 \end{cases}$$

$|x - x^2| < 0$ on $[-1, 0)$

$|x - x^2| > 0$ on $(0, 1)$

$|x - x^2| < 0$ on $(1, 2)$

$$= \int_{-1}^0 -(x-x^2)dx + \int_0^1 (x-x^2)dx + \int_1^2 -(x-x^2)dx = -\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-1}^0 + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 - \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_1^2$$

$$= \frac{(-1)^2}{2} - \frac{(-1)^3}{3} + \frac{1}{2} - \frac{1}{3} - \left(\frac{4}{2} - \frac{8}{3} - \frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{2} + \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{3}} - \left(2 - \frac{7}{3} - \frac{1}{2}\right)$$

$$= 1 - 2 + \frac{7}{3} + \frac{1}{2} = \frac{7}{3} - \frac{1}{2} = \boxed{\frac{11}{6}}$$

$(a+b)^2 = a^2+2ab+b^2$

$\boxed{\int \frac{1}{x}dx = \ln|x|+C}$

(d) $\displaystyle\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left((\sqrt{x})^2 + 2\cancel{\sqrt{x}}\frac{1}{\cancel{\sqrt{x}}} + \left(\frac{1}{\sqrt{x}}\right)^2\right) dx$

$\displaystyle = \int_4^9 \left(x+2+\frac{1}{x}\right) dx = \left[\frac{x^2}{2}+2x+\ln|x|\right]_4^9$

$\displaystyle = \frac{81}{2}+18+\ln|9| - \left(\frac{16}{2}+8+\ln|4|\right)$

$\displaystyle = \boxed{\frac{65}{2}+10+\ln|9|-\ln|4|} = \frac{85}{2}+\ln\left|\frac{9}{4}\right|$

$\boxed{\int \frac{dx}{x^2+1} = \arctan x + C}$

(e) $\displaystyle\int_0^1 \left(x^2+1+\frac{1}{x^2+1}\right) dx = \left(\frac{x^3}{3}+x+\arctan x\right)\Big|_0^1$

$\displaystyle = \frac{1}{3}+1+\underbrace{\arctan 1}_{\pi/4} - \underbrace{\arctan 0}_{0}$

$\displaystyle = \boxed{\frac{4}{3}+\frac{\pi}{4}}$

$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

(f) $\displaystyle\int_{3\pi/4}^{\pi} \sec x \tan x \, dx = \sec x \Big]_{3\pi/4}^{\pi} = \sec\pi - \sec\frac{3\pi}{4}$

$\boxed{(\sec x)' = \sec x \tan x}$

$\displaystyle = -1 + \frac{2}{\sqrt{2}} = -1+\frac{2\sqrt{2}}{2}$

$= \boxed{-1+\sqrt{2}}$

2. If $f(3) = 12$, f' is continuous, and $\displaystyle\int_3^5 f'(x)\,dx = 20$, what is the value of $f(5)$?

$\displaystyle\underbrace{\int_3^5 f'(x)dx}_{20} = f(5) - \underbrace{f(3)}_{12}$

$20 = f(5) - 12$

$\boxed{f(5) = 32}$

$\int f(g(x))g'(x)\,dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right| = \int f(u)\,du$

$\int_a^b f(g(x))g'(x)\,dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \\ a \to g(a) \\ b \to g(b) \end{array} \right| = \int_{g(a)}^{g(b)} f(u)\,du$

3. Find the integral.

(a) $\int e^{\overset{u}{2016x}}\, \underset{\frac{du}{2016}}{dx}$

$\left| \begin{array}{l} u = 2016x \\ du = (2016x)'dx \\ \frac{du}{2016} = \frac{2016\,dx}{2016} \\ dx = \frac{du}{2016} \end{array} \right| = \int e^u \frac{du}{2016} = \frac{1}{2016}\int e^u du$

$= \frac{1}{2016} e^u + C$

$= \boxed{\frac{1}{2016} e^{2016x} + C}$

(b) $\int_{\underbrace{2}_{8\pi}}^{\overbrace{4}^{16\pi}} \sin(\underbrace{4\pi x}_{u})\, \overbrace{dx}^{\frac{du}{4\pi}}$

$\left| \begin{array}{l} u = 4\pi x \\ du = (4\pi x)'dx \\ du = 4\pi\, dx \Rightarrow dx = \frac{du}{4\pi} \\ 2 \longmapsto 4\pi(2) = 8\pi \\ 4 \longmapsto 4\pi(4) = 16\pi \end{array} \right| = \int_{8\pi}^{16\pi} \sin u \frac{du}{4\pi}$

$= \frac{1}{4\pi}\int_{8\pi}^{16\pi} \sin u\, du$

$= -\frac{1}{4\pi}\cos u \Big]_{8\pi}^{16\pi} = -\frac{1}{4\pi}\left(\overbrace{\cos(16\pi)}^{1} - \overbrace{\cos(8\pi)}^{1}\right)$

$= \boxed{0}$

(c) $\int_0^{\pi/2} \cos^7 x \sin(2x)\, dx = \int_0^{\pi/2} \cos^7 x\,(2\sin x\cos x)dx = -2\int_{\underbrace{0}_{1}}^{\overbrace{\pi/2}^{0}} \underbrace{\cos^8 x}_{u^8}\underbrace{(-\sin x\, dx)}_{du}$

$\left| \begin{array}{l} u = \cos x \\ du = -\sin x\, dx \\ 0 \to \cos 0 = 1 \\ \frac{\pi}{2} \to \cos\frac{\pi}{2} = 0 \end{array} \right|$

$\sin(2x) = 2\sin x\cos x$

$\sin x\, dx = -du$

$= -2\int_1^0 u^8\, du = -2\,\frac{u^9}{9}\Big]_1^0$

$= \boxed{\frac{2}{9}}$

(d) $\int_0^1 x^4 e^{9x^5-8}\, dx$

$\left| \begin{array}{l} u = 9x^5 - 8 \\ du = 45x^4 dx \\ x^4 dx = \frac{du}{45} \\ 0 \to 9(0)-8 = -8 \\ 1 \to 9(1)-8 = 1 \end{array} \right| = \int_{-8}^{1} e^u \frac{du}{45} = \frac{1}{45}\int_{-8}^{1} e^u du$

$= \frac{1}{45} e^u \Big]_{-8}^{1} = \boxed{\frac{1}{45}\left(e - e^{-8}\right)}$

(e) $\displaystyle\int \frac{\overbrace{15x^2+50x}^{5(3x^2+10x)}}{x^3+5x^2+8}dx$ $\quad\left| \begin{array}{l} u = x^3+5x^2+8 \\ du = (3x^2+10x)dx \end{array} \right|$

$$= \int \frac{5(3x^2+10x)\,dx}{\underbrace{x^3+5x^2+8}_{=u}}\,du = \int \frac{5du}{u} = 5\int \frac{du}{u}$$

$$= 5\ln|u| + C = \boxed{5\ln|x^3+5x^2+8| + C}$$

(f) $\displaystyle\int \frac{dx}{x\sqrt[3]{\ln x}}$ $\quad\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right| = \displaystyle\int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}du$

$$= \frac{u^{-1/3+1}}{-1/3+1} + C = \frac{u^{2/3}}{2/3} + C = \frac{3}{2}u^{2/3} + C$$

$$= \boxed{\frac{3}{2}(\ln x)^{2/3} + C}$$

(g) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ $\quad\left| \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2}x^{-1/2}dx \\ 2(du) = \left(\frac{1}{2\sqrt{x}}dx\right)2 \\ \frac{1}{\sqrt{x}}dx = 2du \end{array} \right| = \displaystyle\int \cos u\,(2du)$

$$= 2\int \cos u\,du$$

$$= 2\sin u + C$$

$$= \boxed{2\sin\sqrt{x} + C}$$

(h) $\displaystyle\int_0^1 (4x^3+1)(x^4+x)^5dx$ $\quad\left| \begin{array}{l} u = x^4+x \\ du = (4x^3+1)dx \\ 0 \to 0^4+0 = 0 \\ 1 \to 1^4+1 = 2 \end{array} \right| = \displaystyle\int_0^2 u^5du = \frac{u^6}{6}\Big]_0^2$

$$= \frac{2^6}{6} = \frac{64}{6} = \boxed{\frac{32}{3}}$$

(i) $\cancel{\displaystyle\int \frac{dx}{x\sqrt[3]{\ln x}}}$

(j) $\displaystyle\int x^5\sqrt{4+x^3}\,dx = \int \underbrace{(x^3)}_{u-4}(x^2)\underbrace{\sqrt{4+x^3}}_{u}\,dx$ $= \frac{du}{3}$

$u = 4+x^3 \Rightarrow x^3 = u-4$
$du = 3x^2\,dx$
$x^2\,dx = \frac{du}{3}$

$= \int (u-4)\sqrt{u}\,\frac{du}{3} = \frac{1}{3}\int \left(u^{3/2} - 4u^{1/2}\right)du = \frac{1}{3}\left(\frac{u^{3/2+1}}{3/2+1} - 4\frac{u^{1/2+1}}{1/2+1}\right) + C$

$= \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - 4\frac{u^{3/2}}{3/2}\right) + C = \frac{1}{3}\left(\frac{2}{5}u^{5/2} - 4\frac{2}{3}u^{3/2}\right) + C$

$= \boxed{\frac{1}{3}\left(\frac{2}{5}(4+x^3)^{5/2} - \frac{8}{3}(4+x^3)^{3/2}\right) + C}$

(k) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

$u = 1+2x \Rightarrow x = \frac{u-1}{2}$
$du = 2\,dx \Rightarrow dx = \frac{du}{2}$
$0 \to 1+2(0) = 1$
$4 \to 1+2(4) = 9$

$= \displaystyle\int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}}\,\frac{du}{2}$

$= \frac{1}{4}\int_1^9 (u-1)u^{-1/2}\,du = \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right)\Big|_1^9$

$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_1^9 = \frac{2}{4}\left(\frac{1}{3}u^{3/2} - u^{1/2}\right)\Big|_1^9$

$= \frac{1}{2}\left(\frac{1}{3}9^{3/2} - 9^{1/2} - \frac{1}{3}1^{3/2} + 1^{1/2}\right) = \frac{1}{2}\left(\frac{1}{3}(27) - 3 - \frac{1}{3} + 1\right)$

$= \frac{1}{2}\left(7 - \frac{1}{3}\right) = \frac{1}{2}\,\frac{20}{3} = \boxed{\frac{10}{3}}$

4. If f is continuous and $\int_0^8 f(u)du = 3$, find $\int_0^2 x^2 f(x^3)dx$ by making an appropriate substitution.

$\displaystyle\int_0^2 x^2 f(x^3)\,dx$

$u = x^3$
$du = 3x^2\,dx$
$x^2\,dx = \frac{du}{3}$
$0 \to 0^3 = 0$
$2 \to 2^3 = 8$

$= \displaystyle\int_0^8 f(u)\frac{du}{3} = \frac{1}{3}\underbrace{\int_0^8 f(u)\,du}_{3}$

$= \frac{1}{3}(3) = \boxed{1}$