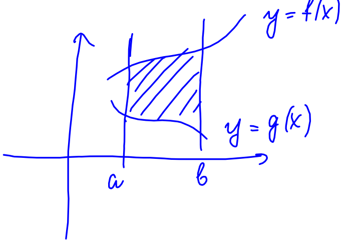
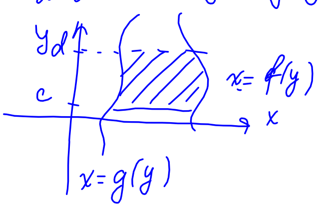
- Area of a region bounded by $y = f(x)$, $y = g(x)$, such that $f(x) \geq g(x)$ on $[a, b]$, $a \leq x \leq b$



[top] − [bottom]

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$

- Area of a region bounded by $x = f(y)$, $x = g(y)$, $c \leq y \leq d$, where $f(y) \geq g(y)$ on $[c, d]$



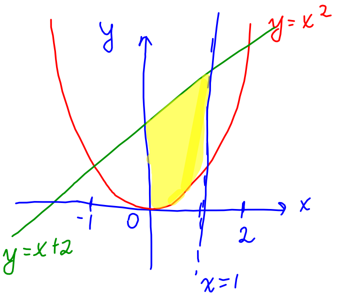
[right] − [left]

$$\boxed{A = \int_c^d [f(y) - g(y)]\,dy}$$

Math 152/172 **WEEK in REVIEW 2** **Spring 2016**

Sections 7.1, 7.2

1. Find the area of the region between $y = x^2$ and $y = x + 2$ from $x = 0$ to $x = 1$.



Points of intersection:

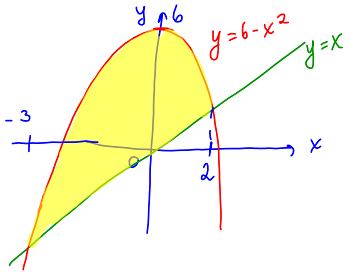
$x^2 = x + 2$ or $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1$ or $x = 2$

$$A = \int_0^1 (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} + 2 - \frac{1}{3} = \boxed{\frac{13}{6}}$$

2. Find the area of the region bounded by the line $y = x$ and the parabola $y = 6 - x^2$.

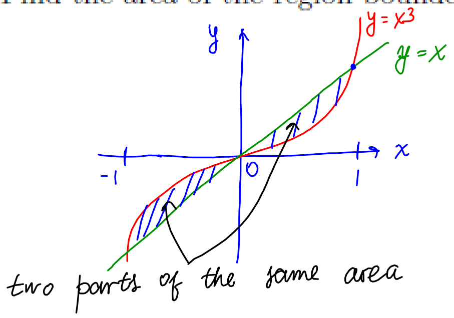


Points of intersection:

$$x = 6 - x^2$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x_1 = -3 \quad \text{or} \quad x_2 = 2$$
$$-3 \le x \le 2$$

$$A = \int_{-3}^{2} (6 - x^2 - x)\,dx = \left[6x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-3}^{2}$$
$$= 6(2) - \frac{8}{3} - \frac{4}{2} - \left(6(-3) - \frac{-27}{3} - \frac{9}{2}\right)$$
$$= 12 - \frac{8}{3} - 2 + 18 - 9 + \frac{9}{2} = \boxed{\frac{125}{6}}$$

3. Find the area of the region bounded by $y = x^3$ and $y = x$.

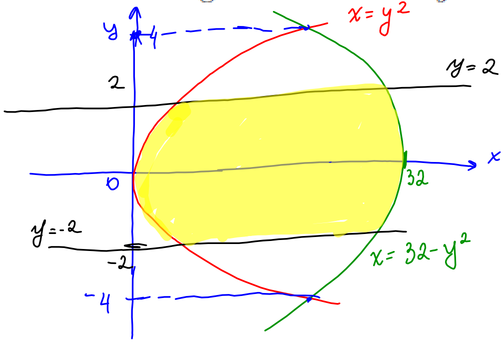


$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$= 2\int_{0}^{1} (x - x^3)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{\frac{1}{2}}$$

4. Find the area of the region between $x = y^2$ and $x = 32 - y^2$ from $y = -2$ to $y = 2$.



Points of intersection:

$y^2 = 32 - y^2$

$2y^2 = 32$

$y^2 = 16$

$y = 4$ or $y = -4$

Integrate for y!

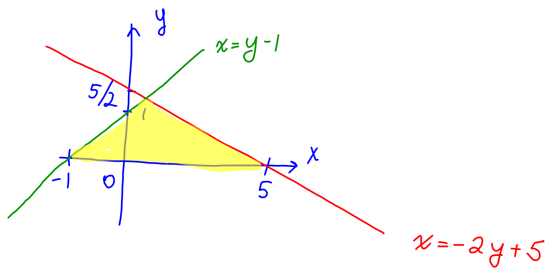
$A = \int_{-2}^{2} \Big[\underbrace{(32 - y^2)}_{\text{[right]}} - \underbrace{y^2}_{\text{[left]}} \Big]\, dy$

$= 2\int_0^2 [32 - 2y^2]\, dy$

$= 2\left[32y - \frac{2y^3}{3}\right]_0^2$

$= \boxed{2\left[64 - \frac{16}{3}\right]}$

5. Find the area of the region between lines $x = -2y + 5$, $x = y - 1$ and $\underset{x\text{-axis}}{y = 0}$.



Point of intersection:

$-2y+5 = y-1 \Rightarrow 3y = 6 \Rightarrow y = 2$

$0 \leq y \leq 2$

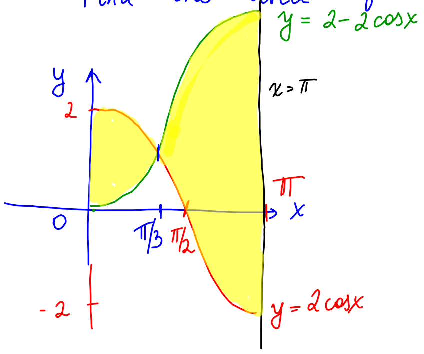
$A = \int_0^2 \left(-2y+5-(y-1)\right) dy$

$= \int_0^2 \left(-3y+6\right) dy = \left[-\frac{3y^2}{2} + 6y\right]_0^2$

$= -6 + 12 = \boxed{6}$

6. ~~Find the area of the region between $x = -y^2$ and $x = y - 2$.~~

Find the area of the region between $y = 2\cos x$, $y = 2 - 2\cos x$, $0 \le x \le \pi$



Point of intersection:

$2\cos x = 2 - 2\cos x$

$4\cos x = 2$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$

$$A = \int_0^{\pi/3} \left(2\cos x - (2 - 2\cos x)\right) dx + \int_{\pi/3}^{\pi} \left((2 - 2\cos x) - 2\cos x\right) dx$$

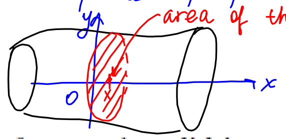
$$= \int_0^{\pi/3} (4\cos x - 2)\, dx + \int_{\pi/3}^{\pi} (2 - 4\cos x)\, dx$$

$$= \left(4\sin x - 2x\right)\Big|_0^{\pi/3} + \left(2x - 4\sin x\right)\Big|_{\pi/3}^{\pi}$$

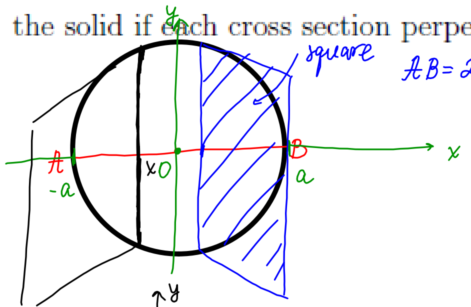
$$= \boxed{4\frac{\sqrt{3}}{2} - \frac{2\pi}{3} + 2\pi - \frac{2\pi}{3} + 4\frac{\sqrt{3}}{2}}$$

$V = \int_a^b A(x)\,dx$, where $A(x)$ is the area of a moving crossectional by a plane perpendicular to the x-axis through $a \le x \le b$.

area of that thing is $A(x)$



7. The base of a certain solid is a circle with diameter AB of length $2a$. Find the volume of the solid if each cross section perpendicular to AB is a square.



$AB = 2a$

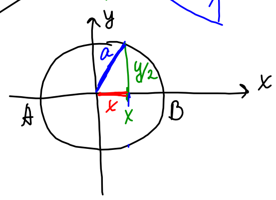
$-a \le x \le a$

area of a square = ?

square of side y.

Area $= y^2$

Express y in terms of x.



$$\left(\frac{y}{2}\right)^2 = a^2 - x^2$$

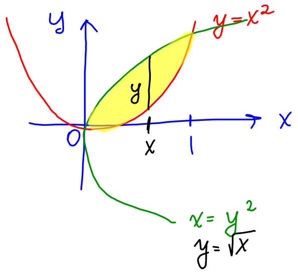
$$\frac{y^2}{4} = a^2 - x^2$$

$$\text{Area} = y^2 = 4(a^2 - x^2)$$

$$V = \int_{-a}^{a} 4(a^2 - x^2)\,dx = \left[4a^2x - \frac{4x^3}{3}\right]_{-a}^{a}$$

$$= \boxed{4a^3 - \frac{4a^3}{3} - \left(-4a^3 - \frac{4(-a^3)}{3}\right)} = \boxed{\frac{16a^3}{3}}$$

8. The base of a certain solid is the region in the xy−plane bounded by the parabolas $y = x^2$ and $x = y^2$. Find the volume of this solid if every cross section perpendicular to the x−axis is a square with base in the xy−plane.



$0 \le x \le 1$

Pick an arbitrary $0 \le x \le 1$
Slice the solid by the plane perpendicular the x-axis through x.
Cross section is the square of side y.

Area = y^2.
Express y in terms of x.

$y = [\text{top}] - [\text{bottom}]$
$= \sqrt{x} - x^2$

area $A(x) = y^2 = (\sqrt{x} - x^2)^2$

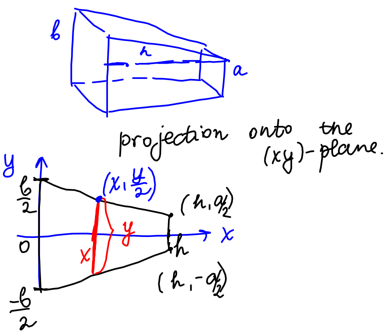
$$V = \int_0^1 (\sqrt{x} - x^2)^2\,dx = \int_0^1 \Big(x - 2\underbrace{(\sqrt{x})(x^2)}_{x^{5/2}} + x^4\Big)\,dx$$

$$= \int_0^1 \left(x - 2x^{5/2} + x^4\right)dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^{5/2+1}}{5/2+1} + \frac{x^5}{5}\right]_0^1 = \left[\frac{x^2}{2} - \frac{2x^{7/2}}{7/2} + \frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \boxed{\frac{1}{2} - \frac{4}{7} + \frac{1}{5}}$$

9. Find the volume of a frustum of a pyramid with square base of side b, square top of side a and height h.



$0 \le x \le h$

Cross section is a square of side y.

Equation of the line through $(0, b/2)$ and $(h, \frac{a}{2})$

$$\text{slope} = \frac{\frac{a}{2} - \frac{b}{2}}{h} = \frac{a-b}{2h}$$

$$\frac{y}{2} - h = \frac{a-b}{2h}\left(x - \frac{a}{2}\right)$$

$$\left(\frac{y}{2} = \frac{a-b}{2h}\left(x - \frac{a}{2}\right) + h\right)(2)$$

$$y = \frac{a-b}{h}\left(x - \frac{a}{2}\right) + 2h$$

$$A(x) = y^2 = \left[\frac{a-b}{h}\left(x - \frac{a}{2}\right) + 2h\right]^2$$

$$= \left(\frac{a-b}{h}\right)^2\left(x - \frac{a}{2}\right)^2 + 2\,\frac{a-b}{h}\left(x - \frac{a}{2}\right)(2h) + 4h^2$$

$$= \left(\frac{a-b}{h}\right)^2\left(x - \frac{a}{2}\right)^2 + 4(a-b)\left(x - \frac{a}{2}\right) + 4h^2$$

$$V = \int_0^h A(x)\,dx = \int_0^h \left[\left(\frac{a-b}{h}\right)^2\left(x - \frac{a}{2}\right)^2 + 4(a-b)\left(x - \frac{a}{2}\right) + 4h^2\right]dx$$

$$= \left(\frac{a-b}{h}\right)^2 \int_0^h \left(x - \frac{a}{2}\right)^2 dx + 4(a-b)\int_0^h \left(x - \frac{a}{2}\right)dx + \int_0^h 4h^2\,dx$$

$u = x - \frac{a}{2}$

$du = dx$

$0 \to -a/2$

$h \to h - \frac{a}{2}$

$$= \left(\frac{a-b}{h}\right)^2 \int_{-a/2}^{h-a/2} u^2\,du + 4(a-b)\int_{-a/2}^{h-a/2} u\,du + 4h^2 x\Big]_0^h$$

$$= \left(\frac{a-b}{h}\right)^2 \frac{u^3}{3}\Big]_{-a/2}^{h-a/2} + 4(a-b)\frac{u^2}{2}\Big]_{-a/2}^{h-a/2} + 4h^3$$

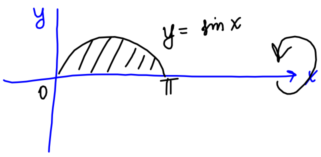
$$= \left(\frac{a-b}{h}\right)^2\left[\frac{\left(h - \frac{a}{2}\right)^3}{3} - \frac{\left(-\frac{a}{2}\right)^3}{3}\right] + 2(a-b)\left[\left(h - \frac{a}{2}\right)^2 - \left(-\frac{a}{2}\right)^2\right] + 4h^3$$

$$= \boxed{\frac{h}{3}\left(a^2 + ab + b^2\right)}$$

disks: $V_x = \pi \int_a^b [f(x)]^2 dx$ or $V_y = \pi \int_c^d [g(y)]^2 dy$

x - axis y - axis

10. Find the volume of the solid which is generated by rotating the region bounded by $y = \sin x$ on $[0, \pi]$ and $y = 0$ about the x−axis.



$V_x = \pi \int_0^{\pi} [\sin x]^2 dx$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

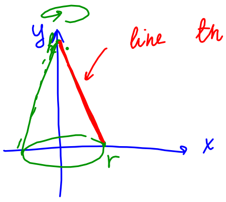
$\boxed{\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C}$

$= \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx = \pi \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \pi \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{\pi}$

$= \pi \, \frac{1}{2} \, \pi = \boxed{\frac{\pi^2}{2}}$

11. Verify the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of the circular cone with base radius r and height h.



line through $(r,0)$ and $(0,h)$

$\frac{x}{r} + \frac{y}{h} = 1$

$y = h\left(1 - \frac{x}{r}\right)$

$y = h - \frac{h}{r}x$

Integrate for $0 \le y \le h$.

$x = r\left(1 - \frac{y}{h}\right)$

$$V = \pi \int_0^h \left[r\left(1 - \frac{y}{h}\right)\right]^2 dy$$

$$= \pi \int_0^h \left(r^2\left(1 - \frac{y}{h}\right)^2\right) dy = \pi r^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy$$

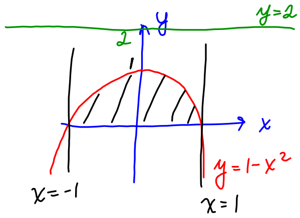
$$= \pi r^2 \left(y - \frac{\cancel{2}y^2}{\cancel{2}h} + \frac{y^3}{3h^2}\right)\Big|_0^h$$

$$= \pi r^2 \left(\cancel{h} - \frac{\cancel{h^2}}{h} + \frac{h^{\cancel{3}}}{3\cancel{h^2}}\right) = \boxed{\frac{1}{3}\pi r^2 h}$$

Washers. $V = \pi \int_{\text{lower limit}}^{\text{upper limit}} \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] \begin{matrix} dx \\ dy \end{matrix}$

12. Find the volume of the solid generated by rotating the region bounded by $y = 1 - x^2$, lines $x = -1$ ~~and~~ $x = 1$ and $y = 0$ about the line $y = 2$.

parallel to the x-axis. integrate for x. $-1 \le x \le 1$



$$V = \pi \int_{-1}^{1} \left([OR]^2 - [IR]^2 \right) dx$$

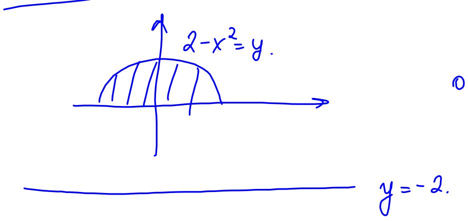
outer radius $= [OR] = 2$

inner radius $= [IR] = 2 - (1 - x^2) = 1 + x^2$

$$V = \pi \int_{-1}^{1} \left(4 - (1+x^2)^2 \right) dx = \pi \int_{-1}^{1} \left(4 - (1 + 2x^2 + x^4) \right) dx$$

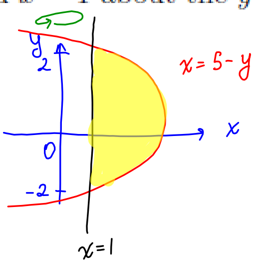
$$= \pi \int_{-1}^{1} (3 - 2x^2 - x^4)\, dx = \pi \left[3x - \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{1}$$

$$= \boxed{\pi \left(3 - \frac{2}{3} - \frac{1}{5} - \left(-3 - \left(-\frac{2}{3}\right) - \left(-\frac{1}{5}\right) \right) \right)} = \boxed{\pi \left(6 - \frac{4}{3} - \frac{2}{5} \right)}$$



$OR = 2 + (2 - x^2)$

13. Determine the volume of the solid obtained by rotating the region bounded by $x = 5 - y^2$ and $x = 1$ about the y-axis.



points of intersection:

$1 = 5 - y^2 \Rightarrow y^2 = 4 \Rightarrow y = -2$ and $y = 2$

$-2 \le y \le 2$

IR = 1

OR = $5 - y^2$

$$V = \pi \int_{-2}^{2} \left[(5-y^2)^2 - 1 \right] dy$$