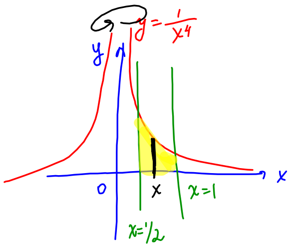
Math 152/172 WEEK in REVIEW 3 Spring 2016

Sections 7.3, 7.4, 7.5

1. Find the volume generated by rotating the region by the given curves.

(a) $y = \dfrac{1}{x^4}$, $x = \dfrac{1}{2}$, $x = 1$, $y = 0$. Rotate about the y-axis. = integrate for x.



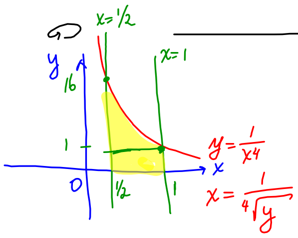
$\frac{1}{2} \le x \le 1$

[radius] = x

[height] = $\frac{1}{x^4}$

$$V_y = 2\pi \int_{1/2}^{1} \underbrace{x}_{\text{radius}} \overbrace{\frac{1}{x^4}}^{\text{height}} dx = 2\pi \int_{1/2}^{1} \frac{1}{x^3}\, dx = 2\pi \int_{1/2}^{1} x^{-3}\, dx$$

$$= 2\pi \left. \frac{x^{-2}}{-2} \right]_{1/2}^{1} = \frac{2\pi}{-2}\left[1 - \frac{1}{(\frac{1}{2})^2} \right] = -\pi(1-4) = \boxed{3\pi}$$



$x = 1/2 \Rightarrow y = \frac{1}{(\frac{1}{2})^4} = 16$

WASHERS.

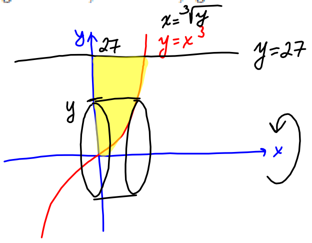
$$V_y = \pi \int_c^d \left([\underset{\text{outer radius}}{OR}]^2 - [\underset{\text{inner radius}}{IR}]^2 \right) dy$$

$$V_y = \pi \left\{ \int_0^1 \left([1]^2 - [\tfrac{1}{2}]^2 \right) dy + \int_1^{16} \left(\left[\tfrac{1}{\sqrt[4]{y}}\right]^2 - [\tfrac{1}{2}]^2 \right) dy \right\}$$

$= \dots = 3\pi$

y-axis

(b) $y = x^3$, $x = 0$, $y = 27$. Rotate about the x-axis. integrate for y.



$0 < y < 27$

$[\text{radius}] = y$

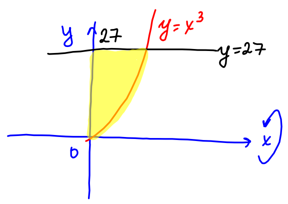
$[\text{height}] = \sqrt[3]{y}$

$$V_x = 2\pi \int_0^{27} [\text{radius}][\text{height}]\, dy$$

$$= 2\pi \int_0^{27} y \sqrt[3]{y}\, dy = 2\pi \int_0^{27} y^{4/3}\, dy$$

$$= 2\pi \left. \frac{y^{4/3+1}}{4/3+1} \right]_0^{27} = 2\pi \left. \frac{y^{7/3}}{7/3} \right]_0^{27}$$

$$= \frac{6\pi}{7}\left((27)^{7/3} - 0\right) = \boxed{\frac{6\pi}{7}(2187)}$$



WASHERS. Integrate for x. $0 \le x \le 3$

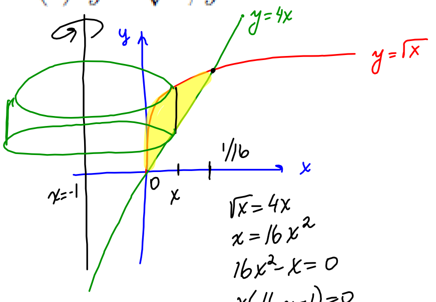
$$V_x = \pi \int_0^3 \left([OR]^2 - [IR]^2\right) dx$$

$[OR] = 27$

$[IR] = x^3$

$$= \pi \int_0^3 \left(27^2 - (x^3)^2\right) dx = \pi \int_0^3 \left(729 - x^6\right) dx = \ldots$$

(c) $y = \sqrt{x}$, $y = 4x$. Rotate about the line $x = -1$.



parallel to the y-axis
integrate for x.

$$\sqrt{x} = 4x$$
$$x = 16x^2$$
$$16x^2 - x = 0$$
$$x(16x-1) = 0$$
$$x = 0 \text{ or } x = \frac{1}{16}$$

[radius] = $1 + x$

[height] = [top] − [bottom]

$= \sqrt{x} - 4x$

$$V_{(x=-1)} = 2\pi \int_0^{1/16} (1+x)(\sqrt{x} - 4x)\,dx$$

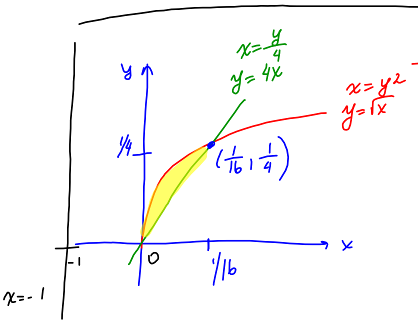
$$= 2\pi \int_0^{1/16} \left(\sqrt{x} - 4x + x\sqrt{x} - 4x^2\right) dx$$

$$= 2\pi \int_0^{1/16} \left(\sqrt{x} - 4x + x^{3/2} - 4x^2\right) dx$$

$$= 2\pi \left(\frac{x^{3/2}}{3/2} - \frac{4x^2}{2} + \frac{x^{5/2}}{5/2} - \frac{4x^3}{3} \right)\Bigg|_0^{1/16}$$

$$= 2\pi \left(\frac{2}{3}\left(\frac{1}{16}\right)^{3/2} - 2\left(\frac{1}{16}\right)^2 + \frac{2}{5}\left(\frac{1}{16}\right)^{5/2} - \frac{4}{3}\left(\frac{1}{16}\right)^3 \right)$$

$$= \boxed{2\pi \left(\frac{2}{3}\,\frac{1}{64} - 2\,\frac{1}{256} + \frac{2}{5}\,\frac{1}{1024} - \frac{4}{3}\,\frac{1}{4096} \right)}$$



WASHERS:

Integrate for y.

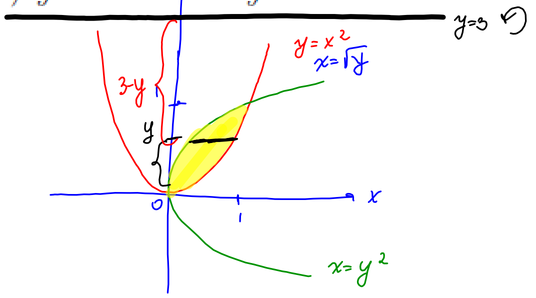
$$0 \le y \le \frac{1}{4}$$

[IR] = $y^2 + 1$

[OR] = $\frac{y}{4} + 1$

$$V = \pi \int_0^{1/4} \left(\left[\frac{y}{4} + 1\right]^2 - \left[y^2 + 1\right]^2 \right) dy$$

(d) $y = x^2$ and $x = y^2$. Rotate about the line $y = 3$.



parallel to the x-axis
integrate for y.

$0 \le y \le 1$

[radius] $= 3 - y$

[height] $= \sqrt{y} - y^2$

$$V = 2\pi \int_0^1 (3-y)(\sqrt{y} - y^2)\, dy$$

$$= 2\pi \int_0^1 (3\sqrt{y} - 3y^2 - y\sqrt{y} + y^3)\, dy$$

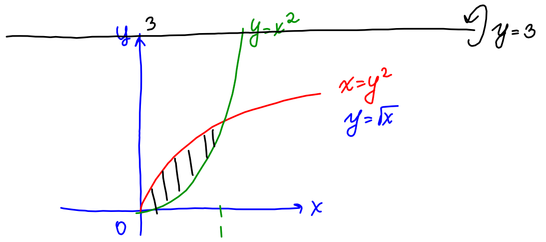
$$= 2\pi \int_0^1 (3\sqrt{y} - 3y^2 - y^{3/2} + y^3)\, dy$$

$$= 2\pi \left(\cancel{3} \frac{y^{3/2}}{\cancel{3}/2} - \frac{\cancel{3}y^3}{\cancel{3}} - \frac{y^{5/2}}{5/2} + \frac{y^4}{4} \right)\Big|_0^1$$

$$= 2\pi \left(2y^{3/2} - y^3 - \frac{2}{5} y^{5/2} + \frac{y^4}{4} \right)\Big|_0^1$$

$$= \boxed{2\pi \left(2 - 1 - \frac{2}{5} + \frac{1}{4} \right)}$$

WASHERS.



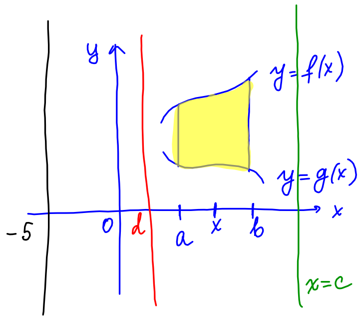
Integrate for x.

$0 \le x \le 1$

[OR] $= 3 - x^2$

[IR] $= 3 - \sqrt{x}$

$$V = \pi \int_0^1 \left[(3 - x^2)^2 - (3 - \sqrt{x})^2 \right] dx$$



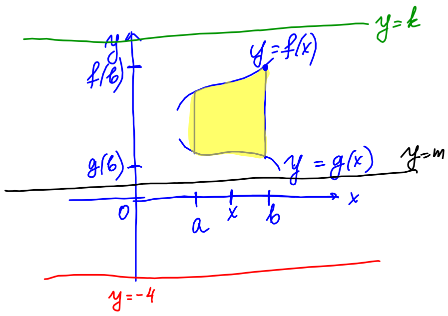
SHELLS

- Rotate about the y-axis. [radius] = x
- Rotate about a line $x = -5$. [radius] = $5 + x$
- Rotate about a line $x = c > b$. [radius] = $c - x$
- Rotate about a line $x = d$, $0 < d < a$, [radius] = $x - d$

[height] = $f(x) - g(x)$

WASHERS:

- Rotate about the x-axis: [IR] = $g(x)$, [OR] = $f(x)$



- Rotate about a line $y = -4$: [IR] = $4 + g(x)$, [OR] = $4 + f(x)$
- Rotate about a line $y = k$ ($k > f(b)$): [IR] = $k - f(x)$, [OR] = $k - g(x)$
- Rotate about a line $y = m$ ($0 < m < g(b)$): [IR] = $g(x) - m$, [OR] = $f(x) - m$

$W=\int_a^b f(x)\,dx$

2. A force of $F(x)=x^4-\sin(4\pi x)+12$ (where x is in meters) acts on an object. What is the work required to move the object from $x=3$ to $x=5$?

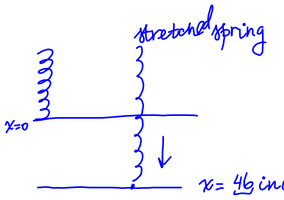
$\boxed{\int \sin ax\,dx = -\frac{1}{a}\cos ax + C}$

$$W=\int_3^5 \left(x^4-\sin(4\pi x)+12\right)dx = \left[\frac{x^5}{5}+\frac{1}{4\pi}\cos(4\pi x)+12x\right]_3^5$$

$$=\frac{5^5}{5}+\frac{1}{4\pi}\underbrace{\cos(20\pi)}_{1}+12(5)-\frac{3^5}{5}-\frac{1}{4\pi}\underbrace{\cos(12\pi)}_{1}-12(3)$$

$$=\boxed{625+60-\frac{243}{5}-36}\ (J)$$

3. If the force required to stretch a spring 3 ft beyond its natural length is 12 lb, how much work is needed to stretch it 46 inches beyond its natural length?



Hook's law

$\boxed{F(x)=kx}$, k is an unknown constant.

$12=k(3) \Rightarrow k=4$

$F(x)=4x$

$x=46$ inches $=\frac{46}{12}$ ft $=\frac{23}{6}$ ft

$$W=\int_0^{23/6} 4x\,dx = \left[4\frac{x^2}{2}\right]_0^{23/6} = 2\left(\frac{23}{6}\right)^2 = \boxed{2\,\frac{529}{36}}\ \text{ft-lb}$$

4. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

natural length $=20$ cm $\to 0$

25 cm $\Rightarrow 25-20=5$ cm $=0.05$ m

30 cm $\Rightarrow 30-20=10$ cm $=0.1$ m

80 cm $\Rightarrow 80-20=60$ cm $=0.6$ m

$f(x)=kx$

$$10=\int_0^{0.05} kx\,dx$$

$$10=k\frac{x^2}{2}\Big]_0^{0.05}$$

$$10=k\,\frac{0.0025}{2}$$

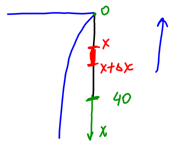
$$k=\frac{20}{0.0025}=\frac{200000}{25}=8000$$

$f(x)=8000x$

$$W=\int_{0.1}^{0.6} 8000x\,dx = 8000\frac{x^2}{2}\Big]_{0.1}^{0.6} = 4000(0.36-0.01)$$

$$=\boxed{4000(0.35)\ (J)}$$

5. A heavy rope 40 ft long, weighs 0.4 lb/ft and hangs over the edge of a tall building. How much work is done in pulling the rope to the top of the building?



$0<x<40$

small part of the rope between x and $x+\Delta x$

$[\text{weight}] = 0.4\,\Delta x$

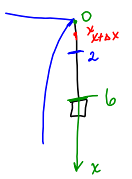
$[\text{distance traveled}] = x$

$$\text{Work done} = \int_0^{40} [\text{weight}][\text{distance traveled}]\,dx$$

$$= \int_0^{40} 0.4x\,dx = 0.4\,\frac{x^2}{2}\Big]_0^{40} = 0.2\,(40)^2$$

$$= 0.2\,(1600) = \boxed{800\ (\text{ft-lb})}$$

6. A uniform cable hanging over the edge of a tall building is 6 m long and weighs 20 kg. If 25 kg weight is attached to the cable, how much work is required to pull 2 m of the cable to the top of the building?



$0\le x\le 6$

$$W = \underbrace{W_1}_{\text{pull the weight up}} + \overbrace{W_2}^{\text{pull the first 2 m of the cable up}} + \underbrace{W_3}_{\text{put the bottom part of the cable up.}}$$

$g = 9.8\ m/sec^2.$

- $W_1 = (25)(2)g = 50g$
- W_2 : $0\le x\le 2$, take a part of the cable between x and $x+\Delta x$

  $[\text{weight}] = \frac{20}{6}\,g\,\Delta x$

  $[\text{distance traveled}] = x$

  $$W_2 = \frac{20}{6}\,g\int_0^2 x\,dx = \frac{10}{3}\,g\,\frac{x^2}{2}\Big]_0^2 = \frac{10}{3}\,g\,(2) = \frac{20}{3}\,g$$

- W_3 : $2\le x\le 6$

  $$W_3 = \frac{20}{6}\,g\,(2) = \frac{20}{3}\,g$$

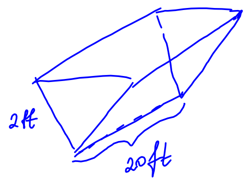
$$\text{Total work} = \boxed{50g + \frac{20}{3}g + \frac{20}{3}g}$$

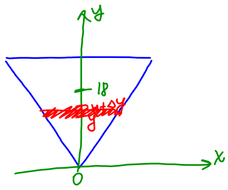
$$W = \overbrace{W_1}^{\text{pull up the weight}} + \underbrace{W_2}_{\text{pull up the cable}}$$

$$W_1 = 25(g)(2) = 50g$$

$$W_2 = \int_2^6 \frac{20}{6}\,g\,x\,dx$$

7. A tank of water is 20 ft long and has a vertical cross section in a shape of an equilateral triangle with sides 2 ft long. The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. The weight of water is 62.5 lb/ft^3.





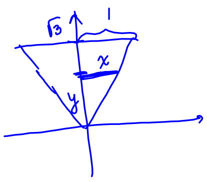
$0 \le y \le \frac{18}{12}$

Pick $0 \le y \le 18$.

Take a "slice" of water between y and $y+\Delta y$.

[weight of the slice] = ?

[distance traveled] = $\sqrt{3} - y$



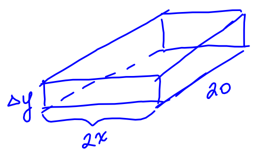
similar triangles

$\frac{\sqrt{3}}{1} = \frac{y}{x}$

$\sqrt{3}x = y \Rightarrow x = \frac{y}{\sqrt{3}}$

height of the triangle $= \sqrt{2^2 - 1^2} =$

slice



$[\text{weight}] = \underbrace{[\text{volume}]}_{(20)(2x)(\Delta y)} (62.5)$

$\text{Volume} = 20\left(\frac{2y}{\sqrt{3}}\right)\Delta y$

$$W = \int_0^{18/12} 20\left(\frac{2y}{\sqrt{3}}\right)(62.5)(\sqrt{3} - y)\,dy$$

$$= \boxed{20\,\frac{2}{\sqrt{3}}(62.5)\int_0^{18/12} y(\sqrt{3} - y)\,dy}$$

8. A spherical tank with radius 8 m is half full of water. The water pumped out of the spout of the top of the tank that is 75 cm high. Find the work done needed to pump out the water through the spout until the water level is 4 m from the bottom. The density of water is 1000 kg/m^3.

9. Determine the average value of the function $f(x) = \dfrac{\sqrt{3}-1}{1+x^2}$ over the interval $[1, \sqrt{3}]$.

10. Find the value(s) a such that the average value of the function $f(x) = 3x^2 - 2x - 3$ over the interval $[a, 0]$ is equal to 2.