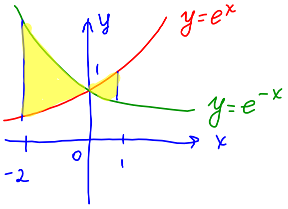
Math 152/172 **WEEK in REVIEW 4** Spring 2016

$$\boxed{\int e^{ax}dx = \frac{1}{a}e^{ax}+c}$$

1. Find the area of the region bounded by $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.



$$A = \int_{-2}^{0} \overbrace{(e^{-x} - e^{x})}^{[top]-[bottom]} dx + \int_{0}^{1} (e^{x} - e^{-x})\, dx$$

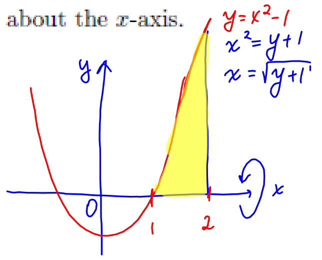
$$= \left[-e^{-x} - e^{x}\right]_{-2}^{0} + \left[e^{x} + e^{-x}\right]_{0}^{1}$$

$$= -e^{0} - e^{0} + e^{-(-2)} + e^{-2} + e + e^{-1} - e^{0} - e^{0}$$

$$= \boxed{e^{2} + e^{-2} + e + e^{-1} - 4}$$

If a region is oriented about the y-axis, then $A = \int_c^d ([right]-[left])\,dy$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 1$, $\underbrace{y = 0}_{x\text{-axis.}}$, $x = 1$, $x = 2$ about the x-axis.



disks:

$$V_x = \pi \int_1^2 (x^2-1)^2\, dx = \pi \int_1^2 (x^4 - 2x^2 + 1)\, dx$$

$$= \pi \left(\frac{x^5}{5} - 2\frac{x^3}{3} + x\right)\Big|_1^2$$

$$= \boxed{\pi\left(\frac{32}{5} - (2)\frac{8}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1\right)}$$

shells. integrate for y. $0 \le y \le 2^2 - 1$

$0 \le y \le 3$

$$V_x = 2\pi \int_0^3 \overbrace{[radius]}^{y}\underbrace{[heigh]}_{\underbrace{[right]}_{2} - \underbrace{[left]}_{\sqrt{y+1}}}\, dy$$

$$= 2\pi \int_0^3 y\left(2 - \sqrt{y+1}\right) dy = 2\pi \int_0^3 2y\, dy - 2\pi \int_0^3 y\sqrt{y+1}\, dy$$

$u = y+1 \Rightarrow y = u-1$

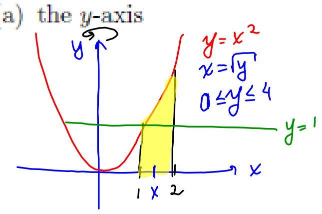
$du = dy$

$0 \mapsto 1,\ 3 \mapsto 4$

$$= 2\pi \left(y^2\right]_0^3 - 2\pi \int_1^4 (u-1)\sqrt{u}\, du = \ldots$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$. $x = 1$, $x = 2$ about

(a) the y-axis



shells:

$$V_y = 2\pi \int_1^2 \underbrace{[\text{radius}]}_{x} \underbrace{[\text{height}]}_{\underbrace{[\text{top}]}_{x^2} - \underbrace{[\text{bottom}]}_{0}} dx$$

$$= 2\pi \int_1^2 x\,(x^2)\,dx$$

$$= 2\pi \int_1^2 x^3\,dx = 2\pi \left.\frac{x^4}{4}\right]_1^2 = 2\pi\left[\frac{16}{4} - \frac{1}{4}\right] = \boxed{\frac{15\pi}{2}}$$

washers: $0 \le y \le 4$

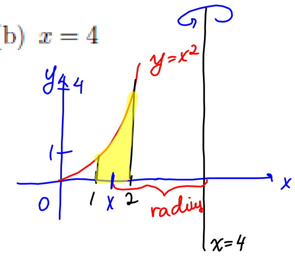
$$V_y = \pi \int_a^b [(\text{right})^2 - (\text{left})^2]\,dy$$

$$V_y = \pi\left[\int_0^1 (2^2 - 1^2)\,dy + \int_1^4 (2^2 - (\sqrt{y})^2)\,dy\right]$$

$$= \pi\left(\int_0^1 3\,dy + \int_1^4 (4 - y)\,dy\right) = \pi\left(3(1-0) + \left(4y - \frac{y^2}{2}\right)\Big|_1^4\right)$$

$$= \boxed{\pi\left(3 + 16 - \frac{16}{2} - 4 + \frac{1}{2}\right)}$$

(b) $x = 4$



shells.

$$V = 2\pi \int_1^2 \underbrace{[\text{radius}]}_{4-x} \underbrace{[\text{height}]}_{x^2} dx$$

$$= 2\pi \int_1^2 (4-x)\,x^2\,dx = 2\pi \int_1^2 (4x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{4x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2$$

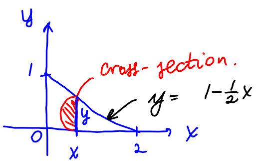
$$= 2\pi\left(\frac{4(8)}{3} - \frac{16}{4} - \frac{4}{3} + \frac{1}{4}\right)$$

$$= \boxed{2\pi\left(\frac{24}{3} - 4 - \frac{4}{3} + \frac{1}{4}\right)}$$

washers:

$$\boxed{V = \pi\left[\int_0^1 \left((\overbrace{3}^{4-1})^2 - (\underbrace{2}_{4-2})^2\right) dy + \int_1^4 \left((4 - \sqrt{y})^2 - 2^2\right) dy\right]}$$

4. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.



$$V = \int_a^b (\text{area of a cross-section})\,dx$$

$$0 \le x \le 2$$

$$A = \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 = \frac{1}{8}\pi y^2$$

Express y in terms of x.

Equation of the line passing through (2,0) and (0,1)

$$\text{slope} = -\frac{1}{2}$$

$$y = -\frac{1}{2}(x-2)$$

$$y = -\frac{1}{2}x + 1 \quad \text{or} \quad y = \left(1 - \frac{1}{2}x\right)$$

$$A = \frac{1}{8}\pi y^2 = \frac{1}{8}\pi\left(1 - \frac{1}{2}x\right)^2$$

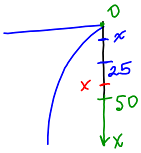
$$V = \frac{\pi}{8}\int_0^2 \left(1 - \frac{1}{2}x\right)^2 dx = \frac{\pi}{8}\int_0^2 \left(1 - x + \frac{1}{4}x^2\right) dx$$

$$= \frac{\pi}{8}\left(x - \frac{x^2}{2} + \frac{x^3}{12}\right)\Big|_0^2$$

$$= \frac{\pi}{8}\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \boxed{\frac{\pi}{12}}$$

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the half rope to the top of the building?

$0 \le x \le 50$



approach 1 Break the rope into two halves:

1) $0 \le x \le 25$

$$W_1 = \int_0^{25} [\underbrace{\text{weight}}_{0.5}][\underbrace{\text{distance traveled}}_{x}]\,dx$$

$$= \int_0^{25} 0.5x\,dx$$

2) $25 \le x \le 50$

$$W_2 = \int_{25}^{50} [\underbrace{\text{weight}}_{0.5}][\underbrace{\text{distance traveled}}_{25}]\,dx$$

$$\text{Total work} = \int_0^{25} 0.5x\,dx + \int_{25}^{50} (0.5)(25)\,dx = \boxed{468.75}\ (\text{ft-lb})$$

approach 2. $W = \int_{25}^{50} 0.5x\,dx = \boxed{468.75}$

6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

$F(x) = kx$

20 cm $\to$ 0

30 cm $\to$ 30 − 20 (cm) = 10 (cm) = .1 (m)

25 cm $\to$ 25 − 20 (cm) = 5 (cm) = .05 (m)

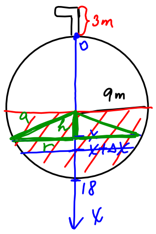
$$25 = k(0.1) \Rightarrow k = 250$$

$$F(x) = 250x$$

$$W = \int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250x\,dx = \frac{250x^2}{2}\Big]_0^{0.05}$$

$$= \left(\frac{250}{2}\, 0.0025\right) = \boxed{0.3125\ (J)}$$

7. A tank in a shape of a sphere of radius 9 m is half full of water. Find the work W required to pump the water out of the spout, if the height of the spout is 3 m.



$9 \le x \le 18$

"Slice" of water between x and $x+\Delta x$

$[\text{weight}] = [\text{volume}](10^3)(9.81)$

$[\text{distance travelled}] = 3+x$

$\text{Volume} = \pi r^2 h \quad (h = \Delta x)$

$= \pi r^2 \Delta x$

$r^2 = 9^2 - h^2 = 9^2 - (x-9)^2 = 81 - (x^2 - 18x + 81)$

$= -x^2 + 18x$

$\text{volume} = \pi(18x - x^2)\Delta x$

$[\text{work done by pumping out the slice}] = \pi(18x-x^2)\Delta x\,(3+x)(10^3)(9.81)$

$W = \pi(10^3)(9.81)\int_9^{18}(18x-x^2)(3+x)\,dx$

$= \pi(10^3)(9.81)\int_9^{18}(54x + 18x^2 - 3x^2 - x^3)\,dx$

$= \pi(10^3)(9.81)\int_9^{18}(54x + 15x^2 - x^3)\,dx$

$= \pi(10^3)(9.81)\left(\frac{54x^2}{2} + \frac{15x^3}{3} - \frac{x^4}{4}\right)\Big|_9^{18} = \ldots$

8. Find the average value of $f(x) = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$f_{ave} = \frac{1}{\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x\,dx$$

$$= \frac{4}{3\pi} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x\,dx$$

$u = \sin x$
$du = \cos x\,dx$
$-\frac{\pi}{2} \to \sin\left(-\frac{\pi}{2}\right) = -1$
$\frac{\pi}{4} \to \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$= \frac{4}{3\pi} \int_{-1}^{\sqrt{2}/2} u^2\,du = \frac{4}{3\pi} \left.\frac{u^3}{3}\right]_{-1}^{\sqrt{2}/2} = \frac{4}{9\pi}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - (-1)^3\right) = \boxed{\frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8} + 1\right)}$$

9. Evaluate the integral

(a) $\int t^2 \cos(1 - t^3)dt$

$u = 1 - t^3$
$du = -3t^2 dt$
$t^2 dt = -\frac{du}{3}$

$$= \int \cos u \left(-\frac{du}{3}\right) = -\frac{1}{3}\int \cos u\,du$$

$$= -\frac{1}{3}\sin u + C = \boxed{-\frac{1}{3}\sin(1-t^3) + C}$$

(b) $\int \frac{x^2}{\sqrt{1-x}}dx$

$u = 1 - x \Rightarrow x = 1 - u$
$du = -dx$

$$= \int \frac{(1-u)^2}{\sqrt{u}}(-du) = -\int \frac{(1-2u+u^2)}{\sqrt{u}}\,du$$

$$= -\int (1-2u+u^2)u^{-1/2}\,du = -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right)du$$

$$= -\left(\frac{u^{1/2}}{1/2} - 2\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2}\right) + C = -2u^{1/2} + \frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C$$

$$= \boxed{-2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C}$$

(c) $\int x^3\sqrt{x^2+5}\,dx = \int x^2 x\sqrt{x^2+5}\,dx$ $\left|\begin{array}{l} u=x^2+5 \Rightarrow x^2=u-5 \\ du=2x\,dx \\ x\,dx=\frac{du}{2}\end{array}\right|$

$= \int (u-5)\sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int (u^{3/2}-5u^{1/2})\,du = \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - 5\frac{u^{3/2}}{3/2}\right)+C$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{10}{3}u^{3/2}\right)+C$

$= \boxed{\frac{1}{5}(x^2+5)^{5/2} - \frac{5}{3}(x^2+5)^{3/2}+C}$

(d) $\int \frac{\sin^3 x}{\sec^4 x}\,dx = \int \sin^3 x\cos^4 x\,dx = \int \sin x\,\overbrace{\sin^2 x}^{1-\cos^2 x}\cos^4 x\,dx = \int \sin x(1-\cos^2 x)\cos^4 x\,dx$ $\left|\begin{array}{l} u=\cos x \\ du=-\sin x\,dx\end{array}\right|$

$= -\int (1-u^2)u^4\,du = -\int (u^4-u^6)\,du = -\frac{u^5}{5}+\frac{u^7}{7}+C$

$= \boxed{-\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+C}$

$\boxed{\int u'v\,dx = uv - \int uv'\,dx}$

(e) $\int x^3 e^{x^2}\,dx = \int x(x^2)e^{x^2}\,dx$ $\left|\begin{array}{l} z=x^2 \\ dz=2x\,dx \\ x\,dx=\frac{dz}{2}\end{array}\right|$ $= \frac{1}{2}\int z e^z\,dz$ $\left|\begin{array}{ll} u=z & u'=1 \\ v'=e^z & v=e^z\end{array}\right|$

$= \frac{1}{2}\left(ze^z - \int e^z\,dz\right) = \frac{1}{2}(ze^z-e^z)+C$

$= \boxed{\frac{1}{2}(x^2e^{x^2}-e^{x^2})+C}$

(f) $\displaystyle\int_0^{\pi/8} \sin^2(2x)\cos^3(2x)\,dx$ $\quad u=2x,\ du=2dx,\ dx=\frac{du}{2}$ $\quad 0\to 0,\ \frac{\pi}{8}\to 2\left(\frac{\pi}{8}\right)=\frac{\pi}{4}$

$$=\frac{1}{2}\int_0^{\pi/4}\sin^2u\cos^3u\,du=\frac{1}{2}\int_0^{\pi/4}\sin^2u\cos u\underbrace{\cos^2u}_{1-\sin^2u}\,du=\frac{1}{2}\int_0^{\pi/4}\sin^2u\cos u\,(1-\sin^2u)\,du$$

$v=\sin u,\ dv=\cos u\,du,\ 0\to\sin 0=0,\ \frac{\pi}{4}\to\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$$=\frac{1}{2}\int_0^{\sqrt{2}/2}u^2(1-u^2)\,du=\frac{1}{2}\int_0^{\sqrt{2}/2}(-u^4+u^2)\,du=\frac{1}{2}\left(-\frac{u^5}{5}+\frac{u^3}{3}\right)\Big|_0^{\sqrt{2}/2}$$

$$=\frac{1}{2}\left(-\frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5+\frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right)=\boxed{\frac{1}{2}\left(-\frac{1}{5}\,\frac{4\sqrt{2}}{32}+\frac{1}{3}\,\frac{2\sqrt{2}}{8}\right)}$$

$\sin^2x=\dfrac{1-\cos 2x}{2}$

$\cos^2x=\dfrac{1+\cos 2x}{2}$

(g) $$\int\sin^2x\cos^4x\,dx=\int\frac{1-\cos 2x}{2}\left(\frac{1+\cos 2x}{2}\right)^2dx=\int\frac{1-\cos 2x}{2}\,\frac{1+2\cos 2x+\cos^2 2x}{4}\,dx$$

$$=\frac{1}{8}\int(1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\,dx=\frac{1}{8}\int(1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x)\,dx$$

$$=\frac{1}{8}\int(1+\cos 2x-\cos^2 2x-\cos^3 2x)\,dx=\frac{1}{8}\left(\int(1+\cos 2x)\,dx-\int\cos^2 2x\,dx-\int\cos^3 2x\,dx\right)$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\int\frac{1+\cos 4x}{2}dx-\int\cos 2x\,\underbrace{(\cos^2 2x)}_{1-\sin^2 2x}\,dx\right)$$

$u=\sin 2x,\ du=2\cos 2x\,dx,\ \cos 2x\,dx=\frac{du}{2}$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)-\int\cos 2x\,(1-\sin^2 2x)\,dx\right)$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x-\int(1-u^2)\frac{du}{2}\right)$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x-\frac{1}{2}\left(u-\frac{u^3}{3}\right)\right)+C=\boxed{\frac{1}{8}\left(\frac{1}{2}x+\cancel{\frac{1}{2}\sin 2x}-\frac{1}{8}\sin 4x-\cancel{\frac{1}{2}\sin 2x}+\frac{1}{6}\sin^3 2x\right)+C}$$

(h) $\displaystyle\int_0^{\pi/4}\tan^4x\sec^2x\,dx$ $\quad u=\tan x,\ du=\sec^2x\,dx,\ 0\to\tan 0=0,\ \frac{\pi}{4}\to\tan\frac{\pi}{4}=1$

$$=\int_0^1u^4\,du=\frac{u^5}{5}\Big]_0^1=\boxed{\frac{1}{5}}$$

(i) $\int \tan x \sec^3 x \, dx = \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right| = \int \sec^2 x (\tan x \sec x) \, dx = \int u^2 du$

$= \frac{u^3}{3} + C = \boxed{\frac{\sec^3 x}{3} + C}$

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$

$\sin 3x \cos x = \frac{1}{2}\left(\sin(3x-x) + \sin(3x+x)\right) = \frac{1}{2}\left(\sin 2x + \sin 4x\right)$

(j) $\int \sin 3x \cos x \, dx = \frac{1}{2}\int(\sin 2x + \sin 4x)\,dx = \boxed{-\frac{1}{2}\left(\frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x\right) + C}$

$\boxed{\int \sin ax \, dx = -\frac{1}{a}\cos ax + C}$