Sections 8.3, 8.4, 8.9

Section 8.3

Table of trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin t, -\pi/2 \leq t \leq \pi/2$	$1 - \sin^2 t = \cos^2 t$
$\sqrt{a^2 + x^2}$	$x = a\tan t, -\pi/2 < t < \pi/2$	$1 + \tan^2 t = \sec^2 t$
$\sqrt{x^2 - a^2}$	$x = a\sec t, 0 \leq t \leq \pi/2$ or $\pi \leq t \leq 3\pi/2$	$\sec^2 t - 1 = \tan^2 t$

1. Evaluate the integral

   (a) $\displaystyle\int \frac{x^2}{\sqrt{4 + x^2}}\, dx$

(b) $\int \frac{x}{\sqrt{6x - x^2}} \, dx$

(c) $\int \frac{dx}{x^2\,\sqrt{25x^2 - 9}}$

Section 8.4

To integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0x^m + b_1x^{m-1} + ... + b_m$:

STEP 1. If f is improper ($n \geq m$), then we must divide P into Q by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax + b$ and *irreducible quadratic factors* (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

STEP 3. Express the proper rational function $\dfrac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Factor	Corresponding fraction
$ax + b$	$\dfrac{A}{ax+b}$
$(ax+b)^r, r > 1$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_r}{(ax+b)^r}$
$ax^2 + bx + c, b^2 - 4ac < 0$	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^r, b^2 - 4ac < 0, r > 1$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

2. Evaluate the integral

   (a) $\displaystyle\int \frac{7}{(x-2)(x+5)}\,dx$

(b) $\int \frac{x^5}{(x-2)^2}\,dx$

(c) $\displaystyle\int \frac{x^2 - 3x + 7}{(x-1)(x^2+1)} dx$

(d) $\displaystyle\int \frac{dx}{(x^2+1)(x^2+x+1)}$

3. Decompose into partial fractions the rational function without computing coefficient of the decomposition:

$$\frac{x-1}{(x+2)^3(x^2-2x+5)^2}$$

4. Compute the following integrals or show that they are diverge.

(a) $\displaystyle\int_e^{\infty} \frac{dx}{x\ \ln^5 x}$

(b) $\displaystyle\int_{-\infty}^{0} (1+x)e^x\ dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{5x^4}{(x^5+3)^3}\,dx$

(d) $\displaystyle\int_{0}^{9} \frac{dx}{\sqrt[3]{x-4}}$

5. Determine whether the given integrals converge or diverge using the Comparison Theorem.

(a) $\displaystyle\int_{0}^{\infty} \frac{dx}{x^7+e^{7x}}$

(b) $\int_{5}^{\infty} \frac{x^2}{x^{5/2} - x} dx$

(c) $\int_{10}^{\infty} \frac{\sin^4(7x)}{x^7} dx$