Math 152/172 **WEEK in REVIEW 6** Spring 2016

Sections 9.3, 9.4, 10.1

- If a curve C is defined by the equations $x = x(t), y = y(t),\quad a \le t \le b$, then $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$
- If a curve C is given by the equation $y = y(x),\quad a \le x \le b$, then $L = \int_a^b \sqrt{1 + [y'(x)]^2}\,dx$
- If a curve C is given by the equation $x = x(y),\quad c \le y \le d$, then $L = \int_c^d \sqrt{1 + [x'(y)]^2}\,dy$

1. Find the length of the curve $x = \cos^3 t, y = \sin^3 t$, $0 \le t \le \pi/2$.

$$L = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$x'(t) = 3\cos^2 t(-\sin t)$

$y'(t) = 3\sin^2 t \cos t$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\,dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t\, \underbrace{(\cos^2 t + \sin^2 t)}_{1}}\,dt$$

$$= \int_0^{\pi/2} 3\cos t \sin t\,dt$$

$u = \sin t$

$du = \cos t\,dt$

$0 \to \sin 0 = 0$

$\pi/2 \to \sin\frac{\pi}{2} = 1$

$$= 3\int_0^1 u\,du = \frac{3u^2}{2}\Big]_0^1 = \boxed{\frac{3}{2}}$$

$$x'(y) = \frac{1}{4y} - y = \frac{1-4y^2}{4y}$$

2. Find the length of the curve $x = \frac{1}{4}\ln y - \frac{1}{2}y^2$ from $y = 1$ to $y = e$.

$$L = \int_1^e \sqrt{1 + [x'(y)]^2}\,dy = \int_1^e \sqrt{1 + \left[\frac{1-4y^2}{4y}\right]^2}\,dy = \int_1^e \sqrt{1 + \frac{(1-4y^2)^2}{16y^2}}\,dy$$

$$= \int_1^e \sqrt{\frac{16y^2 + 1 - 8y^2 + 16y^4}{16y^2}}\,dy = \int_1^e \sqrt{\frac{1 + 8y^2 + 16y^4}{16y^2}}\,dy = \int_1^e \sqrt{\frac{(1+4y^2)^2}{16y^2}}\,dy$$

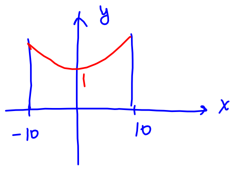
$$= \int_1^e \frac{1+4y^2}{4y}\,dy = \frac{1}{4}\int_1^e (1+4y^2)y^{-1}\,dy = \frac{1}{4}\int_1^e (y^{-1} + 4y)\,dy$$

$$= \frac{1}{4}\left(\ln|y| + \frac{4y^2}{2}\right)_1^e = \frac{1}{4}(\ln e + 2e^2 - \ln 1 - 2) = \boxed{\frac{1}{4}(2e^2 - 1)}$$

3. A wire hanging between two poles (at $x = -10$ and $x = 10$) takes the shape of a catenary with equation

$$y = 2(e^{x/4} + e^{-x/4}),\quad y'(x) = \frac{2}{4}(e^{x/4} - e^{-x/4}) = \frac{1}{2}(e^{x/4} - e^{-x/4})$$

Find the length of the wire.



$$L = \int_{-10}^{10} \sqrt{1 + [y'(x)]^2}\,dx = 2\int_0^{10} \sqrt{1 + [y'(x)]^2}\,dx$$

$$= 2\int_0^{10} \sqrt{1 + [\tfrac{1}{2}\underbrace{(e^{x/4} - e^{-x/4})]^2}}\,dx$$

$$(e^{x/4})^2 - 2e^{x/4}e^{-x/4} + (e^{-x/4})^2 = e^{x/2} - 2 + e^{-x/2}$$

$$\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C}$$

$$= 2\int_0^{10} \sqrt{1 + \tfrac{1}{4}(e^{x/2} - 2 + e^{-x/2})}\,dx = 2\int_0^{10} \sqrt{\frac{4 + e^{x/2} - 2 + e^{-x/2}}{4}}\,dx$$

$$= 2\int_0^{10} \sqrt{\frac{e^{x/2} + 2 + e^{-x/2}}{4}}\,dx = 2\int_0^{10} \sqrt{\frac{(e^{x/4} + e^{-x/4})^2}{4}}\,dx = \cancel{2}\int_0^{10} \frac{e^{x/4} + e^{-x/4}}{\cancel{2}}\,dx$$

$$= \left[4e^{x/4} - 4e^{-x/4}\right]_0^{10} = 4(e^{10/4} - e^{-10/4} - 1 + 1)$$

$$= \boxed{4(e^{5/2} - e^{-5/2})}$$

For rotation about the x-axis, the surface area formulas are:

- if a curve is given as $y = y(x)$, $a \le x \le b$, then $\boxed{S.A. = 2\pi \int_a^b y(x)\sqrt{1 + [y'(x)]^2}dx}$
- if a curve is described as $x = x(y)$, $c \le y \le d$, then $\boxed{S.A. = 2\pi \int_c^d y\sqrt{1 + [x'(y)]^2}dy}$
- if a curve is defined by $x = x(t)$,$y = y(t)$, $a \le t \le b$, then $\boxed{S.A. = \int_a^b y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt}$

For rotation about the y-axis, the surface area formulas are:

- if a curve is given as $y = y(x)$, $a \le x \le b$, then $\boxed{S.A. = 2\pi \int_a^b x\sqrt{1 + [y'(x)]^2}dx}$
- if a curve is described as $x = x(y)$, $c \le y \le d$, then $\boxed{S.A. = 2\pi \int_c^d x(y)\sqrt{1 + [x'(y)]^2}dy}$
- if a curve is defined by $x = x(t)$,$y = y(t)$, $a \le t \le b$, then $\boxed{S.A. = \int_a^b x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt}$

$y' = 2x$

4. The curve $y = x^2$, $0 \le x \le 1$, is rotated about the y-axis. Find the area of the resulting surface.

$$S.A. = 2\pi\int_0^1 x\sqrt{1+[y'(x)]^2}\,dx = 2\pi\int_0^1 x\sqrt{1+4x^2}\,dx$$

$u = 1+4x^2$
$du = 8x\,dx \Rightarrow x\,dx = \frac{du}{8}$
$0 \to 1+4(0)^2 = 1$
$1 \to 1+4(1)^2 = 5$

$$= \frac{2\pi}{8}\int_1^5 \sqrt{u}\,du = \frac{\pi}{4}\left.\frac{u^{3/2}}{3/2}\right]_1^5 = \frac{2\pi}{(4)(3)}\left[5^{3/2}-1\right]$$

$$= \boxed{\frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

$x'(t) = 2\sin(2t)$, $y'(t) = 2+2\cos(2t)$

5. The curve $x = 1 - \cos(2t)$, $y = 2t + \sin(2t)$, $0 \le t \le \pi/4$ is rotated about the x-axis. Find the area of the resulting surface.

$$S.A. = 2\pi\int_0^{\pi/4} y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt = 2\pi\int_0^{\pi/4}(2t+\sin(2t))\sqrt{4\sin^2(2t)+\underbrace{(2+2\cos(2t))^2}_{4+8\cos(2t)+4\cos^2(2t)}}\,dt$$

$$= 2\pi\int_0^{\pi/4}(2t+\sin 2t)\sqrt{4\sin^2(2t)+4+8\cos(2t)+4\cos^2(2t)}\,dt$$

$$= 2\pi\int_0^{\pi/4}(2t+\sin 2t)\sqrt{4(\underbrace{\sin^2(2t)+\cos^2(2t)}_{1}+1+2\cos 2t)}\,dt$$

$\boxed{\frac{1+\cos 2t}{2} = \cos^2 t}$

$$= 2\pi\int_0^{\pi/4}(2t+\sin 2t)\sqrt{4\underbrace{(2+2\cos 2t)}_{4\cos^2 t}}\,dt = 2\pi\int_0^{\pi/4}(2t+\sin 2t)\sqrt{4(4\cos^2 t)}\,dt$$

$$= 2\pi\int_0^{\pi/4}(2t+\sin(2t))\,4\cos t\,dt = 2\pi\left[\int_0^{\pi/4}8t\cos t\,dt + \int_0^{\pi/4}\sin(2t)\cos t\,dt\right]$$

by parts
$u = t$ $u' = 1$
$v' = \cos t$ $v = \sin t$

$$= 2\pi\left(t\sin t\Big]_0^{\pi/4} - \int_0^{\pi/4}\sin t\,dt\right)$$

$$= 2\pi\left(\frac{\pi}{4}\frac{\sqrt{2}}{2} + \cos t\Big]_0^{\pi/4}\right)$$

$$= 2\pi\left(\frac{\sqrt{2}\pi}{8} + \cos\frac{\pi}{4} - \cos 0\right)$$

$$= 2\pi\left(\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - 1\right)$$

$\sin 2t = 2\sin t\cos t$

$$= 2\pi\int_0^{\pi/4}2\sin t\cos t\cos t\,dt$$

$$= 4\pi\int_0^{\pi/4}\sin t\cos^2 t\,dt$$

$u = \cos t$
$du = -\sin t\,dt$
$0 \to \cos 0 = 1$
$\pi/4 \to \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$= 4\pi\int_1^{\sqrt{2}/2}u^2\,du = -4\pi\left.\frac{u^3}{3}\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= -\frac{4\pi}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right] = -\frac{4\pi}{3}\left(\frac{2\sqrt{2}}{8} - 1\right)$$

$$S.A. = \boxed{2\pi\left(\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - 1\right) - \frac{4\pi}{3}\left(\frac{\sqrt{2}}{4} - 1\right)}$$

6. Set up (but don't evaluate) the integral that gives the surface area obtaine by rotating the curve

$$x = \sin(\pi y^2/8), \quad 1 \le y \le 2,$$

$x'(y) = \cos\left(\frac{\pi y^2}{8}\right)\left(\frac{2\pi y}{8}\right) = \frac{\pi y}{4}\cos\left(\frac{\pi y^2}{8}\right)$

(a) about the x-axis

$$S.A. = 2\pi\int_1^2 y\sqrt{1+(x'(y))^2}\,dy = 2\pi\int_1^2 y\sqrt{1+\left[\frac{\pi y}{4}\cos\left(\frac{\pi y^2}{8}\right)\right]^2}\,dy$$

$$= \boxed{2\pi\int_1^2 y\sqrt{1+\frac{\pi^2 y^2}{16}\cos^2\left(\frac{\pi y^2}{8}\right)}\,dy}$$

(b) about the y-axis

$$S.A. = 2\pi\int_1^2 x(y)\sqrt{1+(x'(y))^2}\,dy = \boxed{2\pi\int_1^2 \sin\left(\frac{\pi y^2}{8}\right)\sqrt{1+\frac{\pi^2 y^2}{16}\cos^2\left(\frac{\pi y^2}{8}\right)}\,dy}$$

$x'(t) = a\cos(at), \quad y'(t) = -a\sin(at)$

7. The curve $x = \sin(at), y = \cos(at), 0 \le t \le \frac{\pi}{2a}$ is rotated about the x-axis (here a is an arbitrary positive constant). Find the area of the resulting surface.

$$S.A. = 2\pi\int_0^{\pi/2a} y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

$$= 2\pi\int_0^{\frac{\pi}{2a}} \cos(at)\sqrt{(a\cos(at))^2+(-a\sin(at))^2}\,dt = 2\pi\int_0^{\frac{\pi}{2a}} \cos(at)\sqrt{\underbrace{a^2\cos^2(at)+a^2\sin^2(at)}_{\underbrace{a^2(\cos^2(at)+\sin^2(at))}_{a^2}}}\,dt$$

$$\boxed{\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)+C}$$

$$= 2\pi\int_0^{\frac{\pi}{2a}} a\cos(at)\,dt = 2\pi\,\cancel{a}\,\frac{1}{\cancel{a}}\sin at\Big]_0^{\frac{\pi}{2a}}$$

$$= 2\pi\left[\sin\left(a\frac{\pi}{2a}\right)-\sin 0\right] = \boxed{2\pi}$$

$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ $a_n = f(n),\ n = 1, 2, 3, \ldots$

8. Define the n-th term of the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$ and find its limit.

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{3}{5}$

$\ldots\ \ldots$

$\boxed{a_n = \frac{n}{n+1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\cancel{\frac{1}{n}}^{\,0})} = \boxed{1}$

9. Determine if the given sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{3n^5 - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 - 9n^5}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\boxed{3n^5} - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 \boxed{-9n^5}} = \lim_{n\to\infty} \frac{3\cancel{n^5}}{-9\cancel{n^5}} = \boxed{-\frac{1}{3}}$

(b) $b_n = \dfrac{3n^5 - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 - 9n^5 + 11n^6}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\boxed{3n^5} - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 - 9n^5 + \boxed{11n^6}} = \lim_{n\to\infty} \frac{3n^5}{11n^6} = \lim_{n\to\infty} \frac{3}{11n} = \boxed{0}$

(c) $c_n = \dfrac{12n^7 + 2012}{2012 - 12n^4 - 4n^5 - 9n^6}$

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\boxed{12n^7} + 2012}{2012 - 12n^4 - 4n^5 \boxed{-9n^6}} = \lim_{n\to\infty} \frac{12n^7}{-9n^6} = \lim_{n\to\infty} \frac{12n}{-9} = -\infty$

$\boxed{\text{divergent}}$

10. Determine if the sequence with the given general term ($n \geq 1$) converges or diverges. If it converges, find the limit.

(a) $a_n = \ln(n^2+3) - \ln(7n^2-5)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\ln(n^2+3) - \ln(7n^2-5)\right] = \lim_{n\to\infty}\left(\ln \frac{n^2+3}{7n^2-5}\right) = \ln\Big(\underbrace{\lim_{n\to\infty} \frac{n^2+3}{7n^2-5}}_{1/7}\Big)$$

$$= \boxed{\ln \frac{1}{7} = -\ln 7}$$

The Squeeze Thm. $\{a_n\}, \{b_n\}, \{c_n\}$ such that $a_n \le b_n \le c_n$ for $n > n_0$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.

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(b) $z_n = \dfrac{1}{n^4}\sin\left(\dfrac{1}{n^5}\right)$

$$\underbrace{\frac{-1}{n^4}}_{a_n} \le \underbrace{\frac{\sin \frac{1}{n^5}}{n^4}}_{b_n} \le \underbrace{\frac{1}{n^4}}_{c_n} \quad ; \quad \lim_{n\to\infty}\left(-\frac{1}{n^4}\right) = 0$$

$$\lim_{n\to\infty} \frac{1}{n^4} = 0.$$

By the Squeeze Thm, $\lim_{n\to\infty} \frac{1}{n^4}\sin\left(\frac{1}{n^5}\right) = \boxed{0}$.

(c) $y_n = \dfrac{(-1)^n}{n^3}$, $|y_n| = \dfrac{1}{n^3}$

Thm. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} |y_n| = \lim_{n\to\infty} \frac{1}{n^3} = 0. \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n^3} = \boxed{0}$$

(d) $x_n = \frac{(-1)^n n}{3n+33}$

n is odd $\quad x_n = -\frac{n}{3n+33}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(-\frac{n}{3n+33}\right) = -\frac{1}{3}$

n is even $\quad x_n = \frac{n}{3n+33}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{3n+33} = \frac{1}{3}$ #

$\lim_{n\to\infty} x_n$ $\boxed{DNE}$

11. Assuming that the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$ is convergent, find its limit.

$a_n > 0.$

Let $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$.

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{L} = \lim_{n\to\infty}\frac{1}{2}\left(a_n + \frac{9}{a_n}\right) = \frac{1}{2}\left(\underbrace{\lim_{n\to\infty} a_n}_{L} + \frac{9}{\underbrace{\lim_{n\to\infty} a_n}_{L}}\right)$$

$$L = \frac{1}{2}\left(L + \frac{9}{L}\right)$$

$$2L = L + \frac{9}{L}; \quad 2L^2 = L^2 + 9 \Rightarrow L^2 = 9 \Rightarrow L = \cancel{\pm}3 \quad \boxed{L = 3}$$

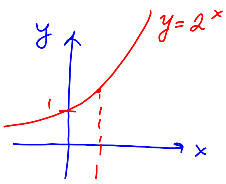
12. Determine whether the given sequence is increasing or decreasing.

(a) $\{\arctan(n)\}_{n=1}^{\infty}$ – $\boxed{\text{increasing}}$

$f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2} > 0$ for all x.

(b) $\{n - 2^n\}_{n=1}^{\infty}$ $\boxed{\text{decreasing}}$



$f(x) = x - 2^x$

$f'(x) = 1 - 2^x \ln 2$, $x \geq 1$

$2^x \ln 2 > 1$ on $[1, \infty)$

$f'(x) = 1 - 2^x \ln 2 < 0$