Math 152/172 **WEEK in REVIEW 7** Spring 2016

Section 10.2, Review for Test 2.

Section 10.2

- Infinite series $\sum_{n=1}^{\infty} a_n$ ($n=1$ for convenience, it can be anything).
- Partial sums: $S_N = \sum_{n=1}^{N} a_n$. Note $S_N = S_{N-1} + a_N$.
- If $\{S_N\}_{N=1}^{\infty}$ is convergent and $\lim_{N\to\infty} S_N = S$ exists as a real number, then the series $\sum_{n=1}^{n} a_n$ is *convergent*. The number s is called the **sum** of the series.
- Series we can sum:
  - Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $-1 < r < 1$
  - Telescoping Series
- THE TEST FOR DIVERGENCE: *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
- **The Test for Divergence cannot be used to prove that a series converges. It can only show a series is divergent.**

1. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

$$S_n = \frac{7n+3}{n+7}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{7n+3}{n+7} = \lim_{n\to\infty} \frac{\cancel{n}\,\frac{7n+3}{n}}{\cancel{n}\,\frac{n+7}{n}} = \lim_{n\to\infty} \frac{\left(7+\cancel{\frac{3}{n}}^{\,0}\right)}{1+\cancel{\frac{7}{n}}^{\,0}} = \boxed{7=S}$$

sum of the series.

The series is convergent.

$$S_1 = a_1,\quad S_2 = \underbrace{a_1}_{S_1}+a_2,\quad S_3 = \underbrace{a_1+a_2}_{S_2}+a_3, \ldots,\quad S_n = \underbrace{a_1+a_2+\ldots+a_{n-1}}_{S_{n-1}}+a_n$$

$$S_n = S_{n-1} + a_n \qquad \boxed{a_n = S_n - S_{n-1}}$$

$$a_n = \underbrace{\frac{7n+3}{n+7}}_{S_n} - \underbrace{\frac{7(n-1)+3}{(n-1)+7}}_{S_{n-1}} = \frac{7n+3}{n+7} - \frac{7n-7+3}{n-1+7} = \frac{7n+3}{n+7} - \frac{7n-4}{n+6}$$

$$= \frac{(7n+3)(n+6) - (7n-4)(n+7)}{(n+7)(n+6)}$$

$$= \frac{\cancel{7n^2}+\cancel{42n}+\cancel{3n}+18 - \cancel{7n^2}-\cancel{49n}+\cancel{4n}+28}{(n+7)(n+6)} = \boxed{\frac{46}{(n+7)(n+6)} = a_n}$$

2. Find the sum of the following series or show they are divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7+5^n}{10^n} = \sum_{n=1}^{\infty}\left[\frac{7}{10^n} + \frac{5^n}{10^n}\right] = \sum_{n=1}^{\infty}\frac{7}{10^n} + \sum_{n=1}^{\infty}\left(\frac{5}{10}\right)^n = 7\sum_{n=\boxed{1}}^{\infty}\left(\frac{1}{10}\right)^n + \sum_{n=\boxed{1}}^{\infty}\left(\frac{1}{2}\right)^n$

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}\ (|r|<1)}$$

$$= 7\underbrace{\sum_{n=1}^{\infty}\frac{1}{10}\left(\frac{1}{10}\right)^{n-1}}_{a=\frac{7}{10},\ r=\frac{1}{10}} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}}_{a=\frac{1}{2},\ r=\frac{1}{2}}$$

$$= \frac{\frac{7}{10}}{1-\frac{1}{10}} + \frac{1/2}{1-1/2} = \frac{7}{9} + 1 = \boxed{\frac{16}{9}}$$

(b) $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)}$ telescoping series.

1) Partial fractions:

$$\frac{8}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} = \frac{A(n+3)+B(n+1)}{(n+1)(n+3)}$$

$$8 = A(n+3)+B(n+1)$$

$n=-3$: $8 = -2B \Rightarrow B = -4$

$n=-1$: $8 = 2A \Rightarrow A = 4$

$$\frac{8}{(n+1)(n+3)} = \boxed{\frac{4}{n+1} - \frac{4}{n+3} = a_n}$$

2) Partial sums.

$$S_1 = a_1 = \frac{4}{1+1} - \frac{4}{1+3} = \frac{4}{2} - \frac{4}{4}$$

$$S_2 = a_1 + a_2 = S_1 + a_2 = \frac{4}{2} - \frac{4}{4} + \frac{4}{2+1} - \frac{4}{2+3} = \frac{4}{2} - \frac{4}{4} + \frac{4}{3} - \frac{4}{5}$$

$$S_3 = S_2 + a_3 = \frac{4}{2} - \frac{4}{4} + \frac{4}{3} - \frac{4}{5} + \frac{4}{3+1} - \frac{4}{3+3}$$

$$= \frac{4}{2} - \cancel{\frac{4}{4}} + \frac{4}{3} - \frac{4}{5} + \cancel{\frac{4}{4}} - \frac{4}{6}$$

$$= \frac{4}{2} + \frac{4}{3} - \frac{4}{5} - \frac{4}{6}$$

$$S_n = \frac{4}{2} + \frac{4}{3} - \frac{4}{n+2} - \frac{4}{n+3}$$

$$\boxed{S_n = \frac{10}{3} - \frac{4}{n+2} - \frac{4}{n+3}}$$

3) Find $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{10}{3} - \frac{4}{n+2} - \frac{4}{n+3}\right) = \boxed{\frac{10}{3} = S}$ sum of the series.

3. Write the repeating decimal $0.\overline{27}$ as a fraction.

$$0.\overline{27} = 0.272727272727$$

$$= 0.27 + 0.0027 + 0.000027 + 0.00000027 + \ldots$$

$$= \frac{27}{100} + \frac{27}{(100)^2} + \frac{27}{(100)^3} + \frac{27}{(100)^4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{27}{(100)^n} = \underbrace{\sum_{n=1}^{\infty} \frac{27}{100}\left(\frac{1}{100}\right)^{n-1}}_{a=\frac{27}{100},\ r=\frac{1}{100}}$$

$$= \frac{\frac{27}{100}}{1-\frac{1}{100}} = \frac{\frac{27}{100}}{\frac{99}{100}} = \boxed{\frac{27}{99}}$$

4. Use the test for Divergence to determine whether the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$

$a_n = \dfrac{n^5}{3(n^4+3)(n+1)}$

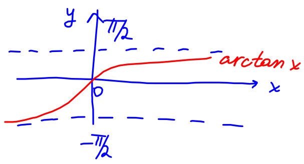
$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^5}{3(n^4+3)(n+1)} = \lim\limits_{n\to\infty} \dfrac{n^5}{3n^4 \frac{n^4+3}{n^4}\, n\, \frac{n+1}{n}}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^5}}{3\cancel{n^5}\left(1+\underbrace{\cancel{\frac{3}{n^4}}}_{0}\right)\left(1+\underbrace{\cancel{\frac{1}{n}}}_{0}\right)} = \dfrac{1}{3} \neq 0$ $\boxed{\text{Divergent}}$

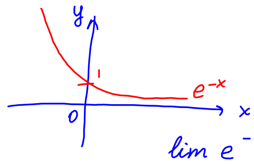
(b) $\displaystyle\sum_{n=1}^{\infty} \arctan n$

$\lim\limits_{n\to\infty} \arctan n = \dfrac{\pi}{2} \neq 0$

$\boxed{\text{Divergent}}$



(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6-e^{-n}}$



$\lim\limits_{x\to\infty} e^{-x} = 0$

$\lim\limits_{n\to\infty} \dfrac{1}{6-e^{-n}} = \dfrac{1}{6-\lim\limits_{n\to\infty} e^{-n}} = \dfrac{1}{6} \neq 0$

$\boxed{\text{Divergent}}$

Review for Test 2. 8.3, 8.4, 8.9, 9.3, 9.4, 10.1, 10.2

1. Evaluate the integral $I = \int (4x^2 - 25)^{-3/2}\, dx$

$(2x)^2 - 5^2$

$2x = 5\sec t$

$x = \frac{5}{2}\sec t \Rightarrow \sec t = \frac{1}{\cos t} = \frac{2x}{5} \Rightarrow \cos t = \frac{5}{2x}$

$dx = \frac{5}{2}\sec t \tan t\, dt$

$(4x^2-25)^{-3/2} = (25\sec^2 t - 25)^{-3/2} = (25\underbrace{(\sec^2 t - 1)}_{\tan^2 t})^{-3/2}$

$= (25\tan^2 t)^{-3/2} = (5\tan t)^{-3}$

$(4x^2-25)^{-3/2} = \frac{1}{125\tan^3 t}$

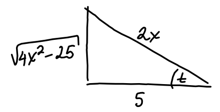
$= \int \frac{\frac{5}{2}\sec t \cancel{\tan t}\, dt}{125\tan^{\cancel{3}\,2} t} = \frac{1}{50}\int \frac{\sec t\, dt}{\tan^2 t}$

$= \frac{1}{50}\int \frac{\frac{1}{\cos t}}{\frac{\sin^2 t}{\cos^2 t}}\, dt = \frac{1}{50}\int \frac{\cos^{\cancel{2}} t}{\cancel{\cos t}\sin^2 t}\, dt$

$= \frac{1}{50}\int \frac{\cos t}{\sin^2 t}\, dt \quad \left|\begin{array}{l} u = \sin t \\ du = \cos t\, dt \end{array}\right| = \frac{1}{50}\int \frac{du}{u^2} = \frac{1}{50}\left(-\frac{1}{u}\right) + C$

$= -\frac{1}{50}\frac{1}{\sin t} + C$

$= \boxed{-\frac{1}{50}\frac{2x}{\sqrt{4x^2-25}} + C}$



$\sin t = \frac{\sqrt{4x^2-25}}{2x}$

$\frac{1}{\sin t} = \frac{2x}{\sqrt{4x^2-25}}$

2. Determine whether the given integral is convergent or divergent.

(a) $\displaystyle\int_1^\infty \frac{4+\cos^4 x}{x}dx$

$\dfrac{4+\cos^4 x}{x}$

$0 \le \cos^4 x \le 1$

$4 \le 4+\cos^4 x \le 4+1$

$\dfrac{4}{x} \le \dfrac{4+\cos^4 x}{x} \le \dfrac{5}{x}$

$\displaystyle\int_1^\infty 4\frac{dx}{x}$ divergent, $\displaystyle\int_1^\infty \frac{5}{x}dx$ - divergent.

$\displaystyle\int_1^\infty \frac{4+\cos^4 x}{x}dx$ diverge by comparison with $\displaystyle\int_1^\infty \frac{4}{x}dx$

(b) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+e^x}dx$

$\dfrac{1}{\sqrt{x}+e^x} \le \dfrac{1}{e^x}$

$\displaystyle\int_0^\infty \frac{1}{e^x}dx = \int_0^\infty e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}\left(-e^{-x}\right)_0^t$

$= \displaystyle\lim_{t\to\infty}\left(\cancel{-e^{-t}}^{\,0}+e^0\right) = 1$ converges.

$\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+e^x}dx$ converges by comparison with $\displaystyle\int_0^\infty \frac{1}{e^x}dx = \int_0^\infty e^{-x}dx$

(c) $\displaystyle\int_0^{2016} \frac{1}{\sqrt{2016-x}}dx = \lim_{t\to 2016^-}\int_0^t \frac{1}{\sqrt{2016-x}}dx = \lim_{t\to 2016^-}\int_0^t (2016-x)^{-1/2}dx$

$= -\displaystyle\lim_{t\to 2016^-} \left.\frac{(2016-x)^{1/2}}{1/2}\right]_0^t = -2\left[\lim_{t\to 2016^-}\left(\cancel{(2016-t)^{1/2}}^{\,0} - (2016)^{1/2}\right)\right] = \boxed{+2(2016)^{1/2}}$

convergent

$y'(x) = \cos x$

3. The curve $y = \sin x$ for $0 \leq x \leq \pi$ is rotated about the x-axis. Set up, *but don't evaluate* the integral for the area of the resulting surface.

$$S.A. = 2\pi \int_0^{\pi} y(x)\sqrt{1 + [y'(x)]^2}\, dx$$

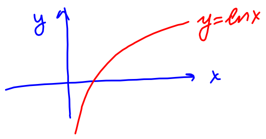
$$= \boxed{2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\, dx}$$

4. Determine if the sequence $\{a_n\}_{n=2}^{\infty}$ is decreasing and bounded:

(a) $a_n = \ln n$

increasing

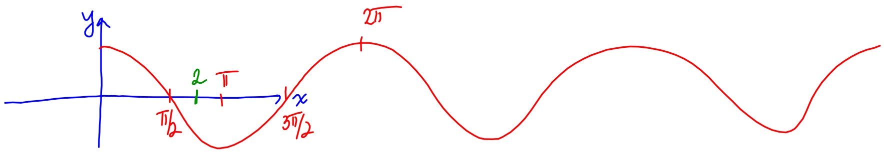
non bounded



(b) $a_n = \cos(n^2)$

$-1 \le \cos(n^2) \le 1$ bounded

not monotonic on $[2,\infty)$

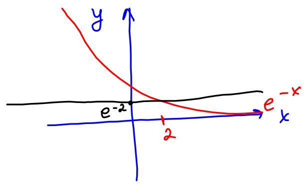


(c) $a_n = e^{-n}$

decreasing on $[2,\infty)$

$0 < e^{-n} \le e^{-2}$

bounded



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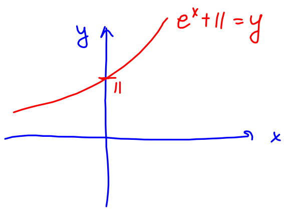
(d) $a_n = e^n + 11$

increasing

non bounded.



(e) $a_n = 1 - \dfrac{1}{n^2}$ $n \geq 2.$

$f(x) = 1 - \frac{1}{x^2}$, $f'(x) = \frac{2}{x^3} > 0$ on $[2, \infty)$

increasing on $[2, \infty)$.

$y'(x) = \frac{1}{2}(e^x - e^{-x})$

5. The curve $y = \frac{1}{2}(e^x + e^{-x})$, $0 \le x \le 1$, is rotated about the x-axis. Find the area of the resulting surface.

$$S.A. = 2\pi \int_0^1 y(x)\sqrt{1 + (y'(x))^2}\,dx$$

$$= \cancel{2}\pi \int_0^1 \frac{1}{\cancel{2}}(e^x + e^{-x})\sqrt{1 + \left[\frac{1}{2}(e^x - e^{-x})\right]^2}\,dx$$

$$= \pi \int_0^1 (e^x + e^{-x})\sqrt{1 + \frac{1}{4}(e^{2x} - 2e^x e^{-x} + e^{-2x})}\,dx$$

$$= \pi \int_0^1 (e^x + e^{-x})\sqrt{1 + \underbrace{\frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}}_{\left[\frac{1}{2}(e^x - e^{-x})\right]^2}}\,dx$$

$$= \pi \int_0^1 (e^x + e^{-x})\sqrt{\underbrace{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}_{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}}\,dx = \pi \int_0^1 (e^x + e^{-x})\sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\,dx$$

$$= \pi \int_0^1 (e^x + e^{-x})\frac{1}{2}(e^x + e^{-x})\,dx = \frac{\pi}{2}\int_0^1 (e^x + e^{-x})^2\,dx$$

$$= \frac{\pi}{2}\int_0^1 (e^{2x} + 2e^x e^{-x} + e^{-2x})\,dx = \frac{\pi}{2}\int_0^1 (e^{2x} + 2 + e^{-2x})\,dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2} - \frac{1}{2} + \frac{1}{2}\right]$$

$$= \boxed{\frac{\pi}{2}\left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right)}$$

$x'(t) = 4t$, $y'(t) = 3t^2$

6. Set up, *but don't evaluate* the integral for the length of the curve $x = 2t^2$, $y = t^3$, $0 \leq t \leq 1$.

$$L = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^1 \sqrt{(4t)^2 + (3t^2)^2}\, dt = \boxed{\int_0^1 \sqrt{16t^2 + 9t^4}\, dt}$$

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

$0 \le x \le \frac{1}{6}$

7. Find length of the curve $y = \frac{1}{\pi}\ln(\sec(\pi x))$ from the point $(0, 0)$ to the point $(\frac{1}{6}, \ln\frac{2}{\sqrt{3}})$.

$$L = \int_0^{1/6} \sqrt{1 + [y'(x)]^2}\, dx$$

$$y'(x) = \frac{1}{\cancel{\pi}} \frac{1}{\cancel{\sec(\pi x)}} \cdot \cancel{\sec(\pi x)} \tan(\pi x) \cancel{\pi} = \tan(\pi x)$$

$$L = \int_0^{1/6} \sqrt{1 + \tan^2(\pi x)}\, dx = \int_0^{1/6} \sqrt{\sec^2(\pi x)}\, dx = \int_0^{1/6} \sec(\pi x)\, dx$$

$$= \frac{1}{\pi} \ln|\sec(\pi x) + \tan(\pi x)| \Big]_0^{1/6}$$

$$= \frac{1}{\pi}\left[\ln\left|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right| - \ln|\sec 0 + \tan 0|\right]$$

$$= \frac{1}{\pi}\left[\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \cancel{\ln|1|}^{\,0}\right]$$

$$= \boxed{\frac{1}{\pi}\ln|\sqrt{3}|} = \boxed{\frac{1}{\pi}\ln\sqrt{3}} = \boxed{\frac{1}{2\pi}\ln 3}$$

8. Use a trigonometric substitution to eliminate the root: $\sqrt{24 - 12x + 2x^2}$.

$$\sqrt{24-12x+2x^2} = \sqrt{2(12\underbrace{-6x+x^2}_{3\cdot 2})}$$

$$= \sqrt{2(x^2-6x+3^2-3^2+12)}$$

$$= \sqrt{2((x^2-6x+9)+3)} = \sqrt{2}\sqrt{(x-3)^2+3}$$

$$= \sqrt{2}\sqrt{3}\sec t$$

$x-3=\sqrt{3}\tan t$

$\sqrt{\quad} = \sqrt{3}\sec t$

9. Determine if the sequence converges or diverges. If converges, find its limit.

(a) $\left\{\dfrac{2016+(-1)^n}{n^{2016}}\right\}_{n=1}^{\infty}$

n is even $\left\{\dfrac{2017}{n^{2016}}\right\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \dfrac{2017}{n^{2016}} = 0$

n is odd $\left\{\dfrac{2015}{n^{2016}}\right\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \dfrac{2015}{n^{2016}} = 0$

match

$\lim\limits_{n\to\infty} \dfrac{2016+(-1)^n}{n^{2016}} = \boxed{0}$ converges.

(b) $\left\{\sqrt{\dfrac{7n+6n^3+n^2}{(n+3)(n^2+8)}}\right\}_{n=4}^{\infty}$. convergent

$$\lim_{n\to\infty} \sqrt{\frac{7n+6n^3+n^2}{(n+3)(n^2+8)}} = \lim_{n\to\infty} \sqrt{\frac{\cancel{n^3}\left(\cancel{\frac{7}{n^2}}^{0}+6+\cancel{\frac{1}{n}}^{0}\right)}{\cancel{n}\left(1+\cancel{\frac{3}{n}}^{0}\right)\cancel{n^2}\left(1+\cancel{\frac{8}{n^2}}^{0}\right)}} = \boxed{\sqrt{6}}$$

(c) $\left\{\dfrac{1}{2}\ln(n^2+2n-4)-\ln(n+6)\right\}_{n=10}^{\infty}$ convergent

$$\lim \left[\frac{1}{2}\ln(n^2+2n-4)-\ln(n+6)\right] = \lim_{n\to\infty}\left[\ln(n^2+2n-4)^{1/2} - \ln(n+6)\right]$$

$$= \lim_{n\to\infty} \ln \frac{\sqrt{n^2+2n-4}}{n+6} = \ln\left[\lim_{n\to\infty} \frac{\sqrt{n^2+2n-4}}{n+6}\right] = \ln 1 = \boxed{0}$$

10. Evaluate the integral $\int \frac{(x-1)^2}{5\sqrt{25-(x-1)^2}}dx$.

$x-1 = 5\sin t$
$x = 5\sin t + 1$
$dx = 5\cos t\, dt$
$\sqrt{25-(x-1)^2} = 5\cos t$

$\sin t = \frac{x-1}{5}$
$t = \arcsin \frac{x-1}{5}$
$\cos t = \frac{\sqrt{25-(x-1)^2}}{5}$

$$= \frac{1}{5}\int \frac{25\sin^2 t}{5\cancel{\cos t}}\, 5\cancel{\cos t}\, dt = 5\int \sin^2 t\, dt = 5\int \frac{1-\cos 2t}{2}\, dt$$

$$= \frac{5}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{5}{2}\left(\arcsin \frac{x-1}{5} - \frac{1}{\cancel{2}}\cancel{2}\sin t \cos t\right) + C$$

$$= \boxed{\frac{5}{2}\left(\arcsin \frac{x-1}{5} - \frac{x-1}{5}\,\frac{\sqrt{25-(x-1)^2}}{5}\right) + C}$$

11. Compute $S = \sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)})$. collapsing series.

$S_1 = e^1 - e^{\frac{1}{2}}$

$S_2 = \underbrace{e^1 - e^{1/2}}_{S_1} + \underbrace{e^{\frac{1}{2}} - e^{1/3}}_{a_2} = e^1 - e^{1/3}$

$S_3 = \underbrace{e^1 - e^{1/3}}_{S_2} + \underbrace{e^{1/3} - e^{1/4}}_{a_3} = e^1 - e^{1/4}$

$S_n = e^1 - e^{\frac{1}{n+1}} = e - e^{\frac{1}{n+1}}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(e - e^{\frac{1}{n+1}}\right) = e - e^{\lim_{n\to\infty} \cancel{\frac{1}{n+1}}^{0}}$

$= \boxed{e - 1}$

12. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3 + 12x^2 + x}{(x^3 - x)(x^3 + 2x^2 - 3x)(x^2 + x + 1)(x^2 + 9)^2}$$

$x^3 - x = x(x^2-1) = x(x-1)(x+1)$

$x^3+2x^2-3x = x(x^2+2x-3)$

$= x(x+3)(x-1)$

$$= \frac{20x^3+12x^2+x}{x(x-1)(x+1)\,x(x+3)(x-1)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)^2(x+1)(x+3)(x^2+x+1)(x^2+9)^2}$$

$$= \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x+1} + \frac{F}{x+3} + \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{x^2+9} + \frac{Kx+L}{(x^2+9)^2}}$$

13. Evaluate the integral $\int \frac{5x^2 + x + 12}{x^3 + 4x} dx$

$$\frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$A=3, \quad B=2, \quad C=1$$

$$\int = \int \left(\frac{3}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4} \right) dx$$

$$= \boxed{3 \ln|x| + \ln|x^2+4| + \frac{1}{2} \arctan \frac{x}{2} + C}$$

14. Assuming that the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{16}{a_n}\right)$ is convergent, find its limit.

Let $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = L$, then

$$L = \frac{1}{2}\left(L + \frac{16}{L}\right)$$

$$L = \frac{1}{2}\,\frac{L^2+16}{L} \Rightarrow 2L^2 = L^2+16$$

$$L^2 = 16 \Rightarrow L = \pm 4.$$

Since $a_1 = 1$, and the sequence is increasing, then $\boxed{L = 4}$

geometric

15. For what values of x the series $\sum_{n=0}^{\infty}(4x-3)^{n+3}$ converges? What is the sum of the series?

The series converges if

$$|4x-3| \le 1$$

$$-1 \le 4x-3 \le 1$$

$$-2 \le 4x \le 4$$

$$\boxed{-\frac{1}{2} \le x \le 1}$$

$$\sum_{n=0}^{\infty} (4x-3)^3 (4x-3)^n$$

geometric, $a = (4x-3)^3$, $r = 4x-3$

$$S = \frac{(4x-3)^3}{1-(4x-3)} = \boxed{\frac{(4x-3)^3}{4-4x}}$$