

10.3 : The Integral and Comparison Tests; Estimating Sums

- **THE TEST FOR DIVERGENCE:** If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.
- **THE INTEGRAL TEST:** Let $\sum a_n$ be a **positive** series. If f is a continuous and decreasing function on $[a, \infty)$ such that $a_n = f(n)$ for all $n \geq a$ then $\sum a_n$ and $\int_a^\infty f(x) dx$ both converge or both diverge.
- **THE COMPARISON TEST:** Suppose that $\sum a_n$ and $\sum b_n$ are series with **nonnegative** terms and $a_n \leq b_n$ for all n .
 1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
 2. If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.
- **LIMIT COMPARISON TEST:** Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms . If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

- The p -series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.
- **REMAINDER ESTIMATE FOR THE INTEGRAL TEST:** If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

Examples.

1. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ is convergent or divergent.
2. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.
3. Determine if the following series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5} \sqrt[3]{8n}}$

(b) $\sum_{n=1}^{\infty} \frac{n^2 + 12}{\sqrt{n^6 + 6}}$

(c) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right)$

$$(d) \sum_{n=1}^{\infty} \frac{5n^5 + e^{-5n}}{6n^6 - e^{-6n}}$$

4. Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ is convergent.

5. (a) If $\sum_{n=1}^{1000} \frac{1}{n^6}$ is used to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

6. Given the series $\sum_{n=1}^{\infty} n^3 e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5 .

10.4 : Other Convergence Tests

- **ALTERNATING SERIES TEST:** If $b_n > 0$, $\lim_{n \rightarrow \infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum (-1)^n b_n$ is convergent.

- **RATIO TEST:** For a series $\sum a_n$ with nonzero terms define $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

1. If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)

2. If $L > 1$ then the series is divergent.

3. If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

- **The Alternating Series Theorem.** If $\sum_{n=1}^{\infty} (-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then $|R_n| \leq b_{n+1}$.

Examples

7. Determine whether the following series converges absolutely, converges but not absolutely, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$, where p is a real parameter.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4 \sqrt{\ln n}}$

(c) $\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)!}$

(d) $\sum_{n=5}^{\infty} \frac{(-1)^{n-1} 7^{n-1}}{4^n}$

(e) $\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{n^2 + n + 1}$

8. Which of the following statements is TRUE?

(a) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) If $\lim_{n \rightarrow \infty} a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(d) If $a_n > 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

9. Given the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^3 e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5 .