Sections 10.3, 10.4.

## 10.3 : The Integral and Comparison Tests; Estimating Sums

- THE TEST FOR DIVERGENCE: If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.
- THE INTEGRAL TEST: Let $\sum a_n$ be a **positive** series. If $f$ is a continuous and decreasing function on $[a,\infty)$ such that $a_n = f(n)$ for all $n \geq a$ then $\sum a_n$ and $\int_a^\infty f(x)\,\mathrm{d}x$ both converge or both diverge.
- THE COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **nonnegative** terms and $a_n \leq b_n$ for all $n$.
  1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
  2. If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.
- LIMIT COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms . If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.
- The $p$-series, $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.
- REMAINDER ESTIMATE FOR THE INTEGRAL TEST: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_n^\infty f(x)\,\mathrm{d}x$$

**Examples.**

1. Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ is convergent or divergent.

2. Find the values of $p$ for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

3. Determine if the following series is convergent or divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5}\,\sqrt[3]{8n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+12}{\sqrt{n^6+6}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^5+e^{-5n}}{6n^6-e^{-6n}}$

4. Find the values of $p$ for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ is convergent.

5. (a) If $\displaystyle\sum_{n=1}^{1000} \frac{1}{n^6}$ is used to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

(b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

6. Given the series $\sum_{n=1}^{\infty} n^3 e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum $s_5$.

## 10.4 : Other Convergence Tests

- ALTERNATING SERIES TEST: If $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum (-1)^n b_n$ is convergent.

- RATIO TEST: For a series $\sum a_n$ with nonzero terms define $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

  1. If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
  2. If $L > 1$ then the series is divergent.
  3. If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

- **The Alternating Series Theorem.** If $\sum_{n=1}^{\infty} (-1)^n b_n$ is a convergent alternating series and you used a partial sum $s_n$ to approximate the sum $s$ (i.e. $s \approx s_n$) then $|R_n| \leq b_{n+1}$.

**Examples**

7. Determine whether the following series converges absolutely, converges but not absolutely, or diverges.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$, where $p$ is a real parameter.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)!}$

(d) $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^{n-1}7^{n-1}}{4^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+n+1}$

8. Which of the following statements is TRUE?

(a) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) If $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(d) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

9. Given the series $\sum_{n=1}^{\infty}(-1)^{n+1}n^3e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum $s_5$.