Math 152/172 **WEEK in REVIEW 9** Spring 2016

Sections 10.5, 10.6

10.5: Power Series

- For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:
  1. There is $R>0$ such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. We call such R the **radius of convergence.**
  2. The series converges only for $x=a$ (then $R=0$).
  3. The series converges for all x (then $R=\infty$).
- We find the radius of convergence using the **Ratio Test.**
- An **interval of convergence** is the interval of all x's for which the power series converges.
- You must check the endpoints $x=a\pm R$ individually to determine whether or not they are in the interval of convergence.

1. For the following series find the radius and interval of convergence.

   (a) $\sum_{n=0}^{\infty} \frac{n^4x^n}{7^n}$

$a_n = \frac{n^4x^n}{7^n}$, $a_{n+1} = \frac{(n+1)^4x^{n+1}}{7^{n+1}}$

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^4x^{n+1}}{7^{n+1}}}{\frac{n^4x^n}{7^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4x^{\cancel{n+1}}}{7^{\cancel{n+1}}}\cdot\frac{\cancel{7^n}}{n^4\cancel{x^n}}\right|$

$= \lim_{n\to\infty}\left|\frac{(n+1)^4x}{7n^4}\right| = \frac{|x|}{|7|}\cancel{\lim_{n\to\infty}\frac{(n+1)^4}{n^4}}^{1} = \left|\frac{x}{7}\right|<1$ or $|x|<\boxed{7}$ $\boxed{R=7}$

$|x|<7$

$-7<x<7$

$x=7$: $\sum_{n=0} \frac{n^4\cancel{7^n}}{\cancel{7^n}} = \sum_{n=0} n^4$ ($\lim_{n\to\infty} n^4=\infty$, diverges by the Divergence Test).

$x=-7$: $\sum_{n=0}\frac{n^4(-7)^n}{7^n} = \sum_{n=0}\frac{n^4(-1)^n\cancel{7^n}}{\cancel{7^n}} = \sum_{n=0} n^4(-1)^n$ (alternating, $b_n=n^4$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^4=\infty$).

Divergent by the Alternating Series Test.

$\boxed{\text{Interval of convergence } (-7,7)}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{8^n (x+4)^{3n}}{n^3+1}$

$$a_n = \frac{8^n (x+4)^{3n}}{n^3+1}, \quad a_{n+1} = \frac{8^{n+1}(x+4)^{3(n+1)}}{(n+1)^3+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{8^{n+1}(x+4)^{3n+3}}{(n+1)^3+1}}{\frac{8^n (x+4)^{3n}}{n^3+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{8^{n+1}}(x+4)^{\cancel{3n+3}}}{((n+1)^3+1)}\cdot\frac{(n^3+1)}{\cancel{8^n}\cancel{(x+4)^{3n}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{8(x+4)^3 (n^3+1)}{(n+1)^3+1}\right| = 8|(x+4)^3| < 1$$

$$\frac{(n^3+1)}{(n+1)^3+1} \to 1$$

$$|(x+4)^3| < \frac{1}{8} \Rightarrow |x+4|^3 < \frac{1}{8}$$

$$|x+4| < \frac{1}{2} \Rightarrow \boxed{R = \frac{1}{2}}$$

$$-\frac{1}{2} < x+4 < \frac{1}{2}$$

$$-\frac{1}{2} - 4 < x < \frac{1}{2} - 4$$

$$-\frac{9}{2} < x < -\frac{7}{2}$$

$x = -\frac{7}{2}$: $\displaystyle\sum_{n=0}^{\infty}\frac{8^n\left(-\frac{7}{2}+4\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty}\frac{8^n\left(\frac{1}{2}\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty}\frac{8^n\left(\frac{1}{8}\right)^n}{n^3+1} = \sum_{n=0}^{\infty}\frac{1}{n^3+1}$ (p-series, $n=3>1$)

convergent

$x = -\frac{9}{2}$: $\displaystyle\sum_{n=0}^{\infty}\frac{8^n\left(-\frac{9}{2}+4\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty}\frac{8^n\left(-\frac{1}{2}\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty}\frac{8^n(-1)^{3n}\left(\frac{1}{8}\right)^n}{n^3+1} = \sum_{n=0}^{\infty}\frac{(-1)^{3n}}{n^3+1}$

alternating series for $b_n = \frac{1}{n^3+1}$

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^3+1} = 0$
- b_n is decreasing, $\left(\frac{1}{n^3+1}\right)' = \frac{-3n^2}{(n^3+1)^2} < 0$ for $n = 0, 1, 2, \ldots$

convergent by the Alternating Series Test.

$\boxed{\left[-\frac{9}{2}, -\frac{7}{2}\right] \text{ interval of convergence}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n(5x-3)^n}{n}$

$$a_n = \frac{(-9)^n(5x-3)^n}{n}, \quad a_{n+1} = \frac{(-9)^{n+1}(5x-3)^{n+1}}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-9)^{n+1}(5x-3)^{n+1}}{n+1}}{\frac{(-9)^n(5x-3)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(-9)^{n+1}(5x-3)^{n+1}}{n+1}\cdot\frac{n}{\cancel{(-9)^n}\cancel{(5x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-9)(5x-3)n}{n+1}\right| = |-9(5x-3)| < 1$$

$$9|5x-3| < 1$$

$$|5x-3| < \frac{1}{9}$$

$$-\frac{1}{9} < 5x-3 < \frac{1}{9}$$

$$-\frac{1}{9}+3 < 5x < \frac{1}{9}+3$$

$$\frac{26}{9} < 5x < \frac{28}{9}$$

$$\frac{26}{45} < x < \frac{28}{45} \implies R = \frac{\frac{28}{45}-\frac{26}{45}}{2} = \boxed{\frac{1}{45} = R}$$

$x = \frac{28}{45}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-9)^n\left(5\left(\frac{28}{45}\right)-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\left(\frac{28}{9}-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\left(\frac{1}{9}\right)^n}{n}$

$\displaystyle = \sum_{n=1}^{\infty}\frac{(-1)^n \cancel{9^n}\cancel{\left(\frac{1}{9}\right)^n}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ – alternating $b_n = \frac{1}{n}$

convergent by the alternating series test.

$x = \frac{26}{45}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-9)^n\left(5\left(\frac{26}{45}\right)-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\left(\frac{26}{9}-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\left(-\frac{1}{9}\right)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

harmonic series divergent

interval of convergence $\boxed{\left(\frac{26}{45}, \frac{28}{45}\right]}$

(d) $\sum_{n=1}^{\infty} \frac{(n+1)!(x-1)^{n+1}}{4^{n+1}}$

$a_n = \frac{(n+1)!\,(x-1)^{n+1}}{4^{n+1}}, \quad a_{n+1} = \frac{(n+2)!\,(x-1)^{n+2}}{4^{n+2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+2)!\,(x-1)^{n+2}}{4^{n+2}}}{\frac{(n+1)!\,(x-1)^{n+1}}{4^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!\,(x-1)^{n+2}\,4^{n+1}}{4^{n+2}\,(n+1)!\,(x-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)!\,(x-1)}{4\,(n+1)!}\right| \overset{(n+2)! = (n+2)(n+1)!}{=\!=} \lim_{n\to\infty}\left|\frac{(n+2)\cancel{(n+1)!}\,(x-1)}{4\cancel{(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)(x-1)}{4}\right| = \begin{cases} 0, & \text{if } x=1 \\ \infty, & \text{if } x\neq 1 \end{cases}$$

converges at $x=1$ only

$$\boxed{\begin{array}{l}\text{Radius of convergence } R=0 \\ \text{Interval of convergence } \{1\}\end{array}}$$

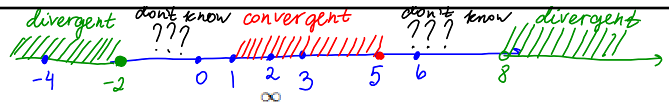
(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-6)^n x^n}{(3n+1)!}$

$$a_n = \frac{(-6)^n x^n}{(3n+1)!}, \quad a_{n+1} = \frac{(-6)^{n+1} x^{n+1}}{(3(n+1)+1)!} = \frac{(-6)^{n+1} x^{n+1}}{(3n+4)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-6)^{n+1} x^{n+1}}{(3n+4)!}}{\frac{(-6)^n x^n}{(3n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-6)^{\cancel{n+1}} x^{\cancel{n+1}}}{(3n+4)!} \cdot \frac{(3n+1)!}{\cancel{(-6)^n} \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-6x\,(3n+1)!}{(3n+4)(3n+3)(3n+2)\underbrace{(3n+1)\cdot\ldots\cdot 1}_{(3n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{-6x}{(3n+4)(3n+3)(3n+2)} \right| = 0 \text{ for all } x.$$

$$\boxed{R = \infty, \text{ interval of convergence } (-\infty, \infty)}$$



2. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges when $x = 5$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following series:

(a) $\sum_{n=0}^{\infty} c_n(-7)^n$ $\boxed{\text{divergent}}$

$x-3=-7$, $x=-7+3=-4$

(b) $\sum_{n=0}^{\infty} c_n 5^n$

$x-3=5$
$x=8$

$\boxed{\text{don't know}}$

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$

$x-3=-3$
$x=0$

$\boxed{\text{don't know}}$

(d) $\sum_{n=0}^{\infty} c_n 3^n$

$x-3=3$
$x=6$

$\boxed{\text{don't know}}$

(e) $\sum_{n=0}^{\infty} c_n(-1)^n$

$x-3=-1$
$x=2$

$\boxed{\text{convergent}}$

10.6: Representation of Functions as Power Series

- Geometric Series Formula:

$$\boxed{\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n}, \quad -1 < x < 1.$$

$$\frac{1}{1-☺} = \sum_{n=0}^{\infty} (☺)^n,$$

convergent if $|☺|<1$

- **Term-by-term Differentiation and Integration of power series:**

If $\boxed{\sum_{n=0}^{\infty} c_n(x-a)^n}$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

– $\boxed{f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}}$

– $\boxed{\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}}$

The radii of convergence of the power series for $f'(x)$ and $\int f(x)\,dx$ are both R.

3. Find a power series representation for the following functions and determine the interval of convergence.

(a) $f(x) = \dfrac{4}{1+x} = 4 \cdot \dfrac{1}{1-(-x)} = 4\sum_{n=0}^{\infty}(-x)^n = 4\sum_{n=0}^{\infty}(-1)^n x^n$

convergent, if $|x|<1$

interval of convergence $(-1,1)$

(b) $f(x) = \dfrac{4}{2+4x} = 4\,\dfrac{1}{2+4x} = 4\,\dfrac{1}{2(1+2x)} = 2\,\dfrac{1}{1+2x} = 2\,\dfrac{1}{1-(-2x)}$

$= 2\sum_{n=0}^{\infty}(-2x)^n = 2\sum_{n=0}^{\infty}(-1)^n 2^n x^n$

$= \sum_{n=0}^{\infty}(-1)^n 2^{n+1} x^n$

interval of convergence $|-2x|<1$

$|x|<\frac{1}{2}$

$\boxed{\left(-\frac{1}{2},\frac{1}{2}\right)}$

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(c) $f(x) = \dfrac{-9x^2}{9-x^4} = -9x^2\,\dfrac{1}{9\left(1-\frac{x^4}{9}\right)} = -x^2 \cdot \dfrac{1}{1-\frac{x^4}{9}} = -x^2\sum_{n=0}^{\infty}\left(\frac{x^4}{9}\right)^n$

$= -\sum_{n=0}^{\infty}\dfrac{x^{4n+2}}{9^n}$

interval of convergence $\left|\frac{x^4}{9}\right|<1$

$|x^4|<9$

$|x|<\sqrt{3}$

$\boxed{(-\sqrt{3},\sqrt{3})}$

(d) $f(x) = \ln(3x+5)$

$f'(x) = \left(\ln(3x+5)\right)' = \frac{3}{3x+5}$

• So the expansion for $\frac{3}{3x+5} = 3\frac{1}{5(1+\frac{3x}{5})} = \frac{3}{5}\cdot\frac{1}{1-(-\frac{3x}{5})}$

$= \sum_{n=0}^{\infty} \frac{3}{5}\left(-\frac{3x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{3}{5}(-1)^n\left(\frac{3}{5}\right)^n x^n$

$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}x^n$

• $\int \frac{3}{3x+5}dx = \ln|3x+5| + C$

$\ln|3x+5| = \int\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}x^n\right)dx$

$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\left(\int x^n dx\right) = \left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}\right] + C$

$\ln(3x+5) = C + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}$

Plug $x=0$: $\ln 5 = C$

$\boxed{\ln(3x+5) = \ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}}$

interval of convergence:

$\left|-\frac{3x}{5}\right| < 1$

$|x| < \frac{5}{3}$

$-\frac{5}{3} < x < \frac{5}{3}$

$x = \frac{5}{3}$: $\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{\left(\frac{5}{3}\right)^{n+1}}{n+1}$

$= \ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ convergent

$x = -\frac{5}{3}$ – divergent.

$\boxed{\left(-\frac{5}{3}, \frac{5}{3}\right]}$ interval of convergence.

(e) $f(x) = x^5\ln(3x+5)$

$x^5\ln(3x+5) = x^5\left[\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}\right]$

$= x^5\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}\cdot x^5}{n+1}$

$= \boxed{x^5\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+6}}{n+1}}$

(f) $f(x) = \frac{x^4}{(1-4x)^2} = x^4\cdot\frac{1}{(1-4x)^2}$

$\int\frac{1}{(1-4x)^2}dx = \left|\begin{array}{l} u = 1-4x \\ du = -4dx \Rightarrow dx = -\frac{du}{4}\end{array}\right| = \int -\frac{du}{4u^2} = \frac{1}{4u} + C$

$= \frac{1}{4(1-4x)} + C$ plug $C=0$.

$\frac{1}{(1-4x)^2} = \left(\frac{1}{4(1-4x)}\right)'$

$\frac{1}{1-4x} = \sum_{n=0}^{\infty}(4x)^n = \sum_{n=0}^{\infty}4^n x^n$

$\frac{1}{4(1+4x)} = \sum_{n=0}^{\infty}\frac{4^n x^n}{4} = \sum_{n=0}^{\infty}4^{n-1}x^n$

$\frac{1}{(1-4x)^2} = \left(\frac{1}{4(1+4x)}\right)' = \sum_{n=1}^{\infty}4^{n-1}nx^{n-1}$

$\frac{x^4}{(1-4x)^2} = x^4\sum_{n=1}^{\infty}4^{n-1}nx^{n-1} = \sum_{n=1}^{\infty}n4^{n-1}x^{n-1}x^4$

$= \boxed{\sum_{n=1}^{\infty}n4^{n-1}x^{n+3}}$

interval of convergence: $|4x| < 1 \Rightarrow |x| < \frac{1}{4}$

$\boxed{\left(-\frac{1}{4}, \frac{1}{4}\right)}$

(g) $f(x) = \arctan(16x^4)$

$$\left(\arctan(16x^4)\right)' = \frac{1}{1+(16x^4)^2}(16\cdot 4\cdot x^3) = \frac{64x^3}{1+256x^8}$$

• do the expansion for $\frac{64x^3}{1+256x^8} = 64x^3 \cdot \frac{1}{1-(-256x^8)}$

$$= 64x^3 \sum_{n=0}^{\infty}(-256x^8)^n = \underbrace{64}_{2^6} x^3 \sum_{n=0}^{\infty}(-1)^n \underbrace{(256)^n}_{2^8} x^{8n} = 2^6 x^3 \sum_{n=0}^{\infty}(-1)^n 2^{8n} x^{8n}$$

$$= \sum_{n=0}^{\infty}(-1)^n 2^{8n+6} x^{8n+3}$$

• $\arctan(16x^4) = \int \frac{64x^3}{1+256x} dx = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\left(\int x^{8n+3} dx\right)$

$$= C + \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\frac{x^{8n+4}}{8n+4}$$

Plug $x=0$: $\arctan 0 = C \Rightarrow C=0$.

$$\boxed{\arctan(16x^4) = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\frac{x^{8n+4}}{8n+4}}$$

interval of convergence : $|16x^4|<1$

$|x|<\frac{1}{2}$

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right]}$

4. Express the integral $\int_{-0.5}^{0}\frac{dx}{1-x^7}$ as a power series.

$$\frac{1}{1-x^7} = \sum_{n=0}^{\infty}(x^7)^n = \sum_{n=0}^{\infty} x^{7n}$$

$$\int_{-0.5}^{0}\frac{dx}{1-x^7} = \int_{-0.5}^{0}\left(\sum_{n=0}^{\infty} x^{7n}\right)dx = \sum_{n=0}^{\infty}\left(\int_{-0.5}^{0} x^{7n}dx\right)$$

$$= \sum_{n=0}^{\infty}\left(\left[\frac{x^{7n+1}}{7n+1}\right]_{-0.5}^{0}\right) = -\sum_{n=0}^{\infty}\frac{(-0.5)^{7n+1}}{7n+1}$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^{7n+1}(0.5)^{7n+1}}{7n+1} = \sum_{n=0}^{\infty}\frac{(-1)\cdot(-1)^{7n+1}(0.5)^{7n+1}}{7n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{7n+2}(0.5)^{7n+1}}{7n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{7n}\cdot(-1)^2(0.5)^{7n+1}}{7n+1}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^{7n}(0.5)^{7n+1}}{7n+1}}$$