Sections 10.7, 10.9, Review for Test 3

10.7: Taylor and Maclaurin Series

- **The Taylor series for $f(x)$ about $x = a$:**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
- **The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
- *Known Mclaurin series and their intervals of convergence you must have memorized:*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad (1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad (-\infty,\infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad (-\infty,\infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad (-\infty,\infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1,1]$$

Examples.

1. Find Maclaurin series for the following functions:
   (a) $f(x) = \sin^2 x$
   (b) $x + 3x^2 + xe^{-x}$
2. Express $\displaystyle\int \frac{\sin(3x)}{x}\,dx$ as an infinite series.
3. Find the sum of the series:
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{2n+1}$

4. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\, dx$ with error less than 10^{-3}.

10.9: Applications of Taylor Polynomials

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

Examples.

5. Find the fourth-degree Taylor polynomial of $f(x) = \dfrac{1}{2+6x}$ centered at $a = 0$.

6. Find the third-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$.

7. Find the second degree Taylor Polynomial for $f(x) = \ln x$ at $a = 3$.

Review for Test 3, covering 10.2–10.7, 10.9

8. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

9. Find the sum of the following series or show they are divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7+5^n}{10^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)}$

10. Write the repeating decimal $0.\overline{27}$ as a fraction.

11. Use the test for Divergence to determine whether the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6-e^{-n}}$

12. Which of the following series converges absolutely?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^2\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\ln(n+1)}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$

13. Suppose that the power series $\displaystyle\sum_{n=1}^{\infty} c_n(x-4)^n$ has the radius of convergence 4. Consider the following pair of series:

$$(I) \quad \sum_{n=1}^{\infty} c_n 5^n \qquad (II) \quad \sum_{n=1}^{\infty} c_n 3^n.$$

Which of the following statements is true?

(a) (I) is convergent, (II)is divergent

(b) Neither series is convergent

(c) Bith series are convergent

(d) (I) is divergent, (II) is convergent

(e) no conclusion can be drawn about either series.

14. Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges. Then find un upper bound on the error in using s_{10} to approximate the series. (Note that $\ln 2 > 1/2$.)

15. If we represent $\dfrac{x^2}{4+9x^2}$ as a power series centered at $a = 0$, what is the associated radius of convergence?

16. Find the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(3x-1)^n}{n}$.

17. Which of the following statements is TRUE?

(a) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(c) If $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(d) If $a_n > 0$ for $n \geq 1$ and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

18. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$ converges to s. Use the Alternating Series Theorem to estimate $|s - s_6|$.