

10.7: Taylor and Maclaurin Series

- The Taylor series for $f(x)$ about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- The Maclaurin series is the Taylor series about $x = 0$ (i.e. $a=0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

- Known Maclaurin series and their intervals of convergence you must have memorized:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad (1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad [-1, 1]$$

Examples.

1. Find Maclaurin series for the following functions:

$$(a) f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n}}{(2n)!}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x = \boxed{\frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} \cdot x^{2n}}{(2n)!}}$$

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$$(b) x + 3x^2 + xe^{-x}$$

$$(-x)^n = (-1)^n \cdot x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{n!}$$

$$xe^{-x} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} = x - x^2 + \frac{x^3}{2!} - \dots$$

$$x + 3x^2 + xe^{-x} = \underline{x + 3x^2} + \underline{x - x^2} + \frac{x^3}{2!} - \frac{x^4}{3!} + \dots$$

$$= 2x + 2x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \dots = \boxed{2x + 2x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n x^{n+1}}{n!}}$$

2. Express $\int \frac{\sin(3x)}{x} dx$ as an infinite series.

$$\begin{aligned} \frac{\sin 3x}{x} &= \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1-1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n}}{(2n+1)!} \end{aligned}$$

$$\int \frac{\sin 3x}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} \left(\int x^{2n} dx \right) = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} \frac{x^{2n+1}}{2n+1}$$

3. Find the sum of the series:

$$(a) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n} (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$(b) \sum_{n=0}^{\infty} \frac{7^n}{n!} = e^7$$

$$(c) \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3(2n+1)}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \arctan(x^3)$$

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4. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2} dx$ with error less than 10^{-3} .

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

$$\int_0^{0.5} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^{0.5} x^{2n+2} dx \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+3}}{2n+3} \Big|_0^{0.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\left(\frac{1}{2}\right)^{2n+3}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3} n! (2n+3)}$$

$$= \boxed{\frac{1}{2^3 \cdot 3}} - \boxed{\frac{1}{2^5 \cdot 5}} + \boxed{\frac{1}{2^7 \cdot 2! \cdot 7}} < 10^{-3} = \boxed{\frac{1}{24} - \frac{1}{160}}$$

$$T_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

10.9: Applications of Taylor Polynomials

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \underbrace{\sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n}_{\substack{T_N(x) \\ N\text{-th degree} \\ \text{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}_{\substack{R_N(x) \\ \text{Remainder}}}$$

Examples.

5. Find the fourth-degree Taylor polynomial of $f(x) = \frac{1}{2+6x}$ centered at $a = 0$,

$$\frac{1}{2+6x} = \frac{1}{2(1+3x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-3x)^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n 3^n x^n$$

$$T_4(x) = \boxed{\frac{1}{2} (1 - 3x + 9x^2 - 27x^3 + 81x^4)}$$

6. Find the **third**-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$.

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$f(x) = \sqrt[3]{x} = x^{1/3}$	$f(1) = 1$
$f'(x) = \frac{1}{3}x^{-2/3}$	$f'(1) = \frac{1}{3}$
$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3}$	$f''(1) = -\frac{2}{9}$
$f'''(x) = -\frac{2}{9}\left(-\frac{5}{3}\right)x^{-8/3}$	$f'''(1) = \frac{10}{27}$

$$T_3(x) = 1 + \frac{1}{3}(x-1) + \left(-\frac{2}{9}\right) \cdot \frac{1}{2}(x-1)^2 + \frac{10}{27} \cdot \frac{1}{6}(x-1)^3$$

$$= \boxed{1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3}$$

7. Find the **second** degree Taylor Polynomial for $f(x) = \ln x$ at $a = 3$.

$$T_2(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2$$

$f(x) = \ln x$	$f(3) = \ln 3$
$f'(x) = \frac{1}{x}$	$f'(3) = \frac{1}{3}$
$f''(x) = -\frac{1}{x^2}$	$f''(3) = -\frac{1}{9}$

$$T_2(x) = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{9} \cdot \frac{1}{2}(x-3)^2$$

$$= \ln 3 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2$$

Review for Test 3, covering 10.2–10.7, 10.9

8. Given a series whose partial sums are given by $s_n = (7n + 3)/(n + 7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

$$\underbrace{S_{n-1} + a_n = S_n}_{\text{definition}} = \frac{7n+3}{n+7}, \quad S_{n-1} = \frac{7(n-1)+3}{(n-1)+7} = \frac{7n-7+3}{n+6} = \frac{7n-4}{n+6}$$

$$a_n = S_n - S_{n-1} = \frac{7n+3}{n+7} - \frac{7n-4}{n+6}$$

$$= \frac{(7n+3)(n+6) - (7n-4)(n+7)}{(n+6)(n+7)}$$

$$= \frac{\cancel{7n^2} + 42n + 3n + 18 - (\cancel{7n^2} + 49n - 4n - 28)}{(n+6)(n+7)}$$

$$= \frac{45n + 18 - 45n + 28}{(n+6)(n+7)} = \boxed{\frac{46}{(n+6)(n+7)}}$$

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{7n+3}{n+7} = 7 \quad \boxed{\text{convergent, } S=7}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

9. Find the sum of the following series or show they are divergent:

$$\begin{aligned} \text{(a)} \quad \sum_{n=1}^{\infty} \frac{7+5^n}{10^n} &= \sum_{n=1}^{\infty} \frac{7}{10^n} + \sum_{n=1}^{\infty} \frac{5^n}{10^n} = \sum_{n=1}^{\infty} 7 \cdot \left(\frac{1}{10}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{10}\right)^n \\ &= \sum_{n=1}^{\infty} 7 \left(\frac{1}{10}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 7 \cdot \frac{1}{10} \left(\frac{1}{10}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} \\ &\qquad\qquad\qquad a = \frac{7}{10}, r = \frac{1}{10} < 1 \qquad\qquad\qquad a = \frac{1}{2}, r = \frac{1}{2} < 1 \\ &= \frac{7/10}{1-1/10} + \frac{1/2}{1-1/2} = \frac{7/10}{9/10} + \frac{1/2}{1/2} = \frac{7}{9} + 1 = \boxed{\frac{16}{9}} \end{aligned}$$

$$\text{(b)} \quad \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)} \quad \text{collapsing series.}$$

Partial fractions: $\frac{8}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} = \frac{A(n+3) + B(n+1)}{(n+1)(n+3)}$

$$8 = A(n+3) + B(n+1)$$

$$n = -3: \quad 8 = -2B \Rightarrow \boxed{B = -4}$$

$$n = -1: \quad 8 = 2A \Rightarrow \boxed{A = 4}$$

$$a_n = \frac{8}{(n+1)(n+3)} = 4 \left[\frac{1}{n+1} - \frac{1}{n+3} \right]$$

Partial sums: $S_1 = a_1 = 4 \left(\frac{1}{2} - \frac{1}{4} \right)$

$$\begin{aligned} S_2 &= S_1 + a_2 = 4 \left(\frac{1}{2} - \frac{1}{4} \right) + 4 \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= 4 \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \right) \end{aligned}$$

$$\begin{aligned} S_3 &= S_2 + a_3 = 4 \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \right) + 4 \left(\frac{1}{4} - \frac{1}{6} \right) \\ &= 4 \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6} \right) \end{aligned}$$

$$S_n = 4 \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 4 \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = 4 \left(\frac{1}{2} + \frac{1}{3} \right) = 4 \cdot \frac{5}{6} = \boxed{\frac{10}{3}}$$

10. Write the repeating decimal $0.\overline{27}$ as a fraction.

$$0.\overline{27} = 0.27272727\dots$$

$$= 0.27 + 0.0027 + 0.000027 + \dots$$

$$= \frac{27}{100} + \frac{27}{10^4} + \frac{27}{10^6} + \dots$$

$$= \frac{27}{100} \left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots \right) = \frac{27}{100} \sum_{n=0}^{\infty} \left(\frac{1}{10^2} \right)^n = \frac{27}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100} \right)^n$$

$$= \frac{27}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{27}{100} \cdot \frac{100}{99} = \frac{27}{99} = \boxed{\frac{3}{11}}$$

D.T. if $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

11. Use the test for Divergence to determine whether the series diverges.

$$(a) \sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$$

$$a_n = \frac{n^5}{3(n^4+3)(n+1)}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^5}{3(n^4+3)(n+1)} = \lim_{n \rightarrow \infty} \frac{n^5}{3n^4(1+\frac{3}{n^4})n(1+\frac{1}{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{n^5}{3n^5} = \frac{1}{3} \neq 0$$

divergent by D.T.

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}, \quad a_n = \frac{(-1)^n}{n\sqrt{n}}, \quad |a_n| = \frac{1}{n\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n\sqrt{n}} = 0$$

Thm. if $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n\sqrt{n}} = 0$$

D.T. fails

$$(c) \sum_{n=1}^{\infty} \frac{1}{6 - e^{-n}}$$

$$a_n = \frac{1}{6 - e^{-n}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{6 - e^{-n}} = \frac{1}{6} \neq 0$$

divergent by D.T.

12. Which of the following series converges absolutely?

$$(a) \sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^2 \sqrt{n}} = \sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^{5/2}}$$

$$\sum_{n=1}^{\infty} \left| \frac{\sin(\pi^3 n^2)}{n^{5/2}} \right|$$

$$\begin{aligned} |\sin(\pi^3 n^2)| &\leq 1 \\ -1 &\leq \frac{\sin(\pi^3 n^2)}{n^{5/2}} \leq \frac{1}{n^{5/2}} \end{aligned}$$

compare with $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ - $p = \frac{5}{2} > 1$ convergent.

$\sum_{n=1}^{\infty} \frac{|\sin(\pi^3 n^2)|}{n^{5/2}}$ is convergent by C.T., thus $\sum_{n=1}^{\infty} \frac{\sin(\pi^3 n^2)}{n^{5/2}}$ is absolutely convergent

$$|\sin n| \leq 1$$

$$|\cos n| \leq 1$$

$$|\arctan n| \leq \frac{\pi}{2}$$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ not absolutely convergent.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[4]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \quad p = 1/4 < 1 \text{ divergent.}$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$ - do the Alternating Series Test.

$$b_n = \frac{1}{n^{1/4}}, \quad \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n^{1/4}} = 0$$

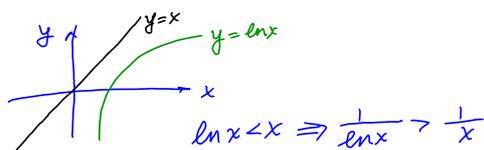
$$b_{n+1} = \frac{1}{(n+1)^{1/4}} \quad \begin{array}{l} n+1 > n \\ (n+1)^{1/4} > n^{1/4} \end{array}$$

$$\frac{1}{(n+1)^{1/4}} < \frac{1}{n^{1/4}} \Rightarrow b_{n+1} < b_n$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$ is convergent, but not absolutely convergent

$$(c) \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$



$\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ - harmonic, divergent.

not absolutely convergent.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ - do A.S.T.

$$b_n = \frac{1}{\ln n}, \quad \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

$\ln n$ is increasing, $\frac{1}{\ln n}$ is decreasing on $[2, \infty)$

$$b_{n+1} < b_n$$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by A.S.T.

convergent, but not absolutely convergent.

$$(d) \sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$$

$$(n+1)! = (n+1)n!$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Ratio Test $a_n = \frac{n^n}{(n!)^2}$, $a_{n+1} = \frac{(n+1)^{n+1}}{[(n+1)!]^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{[(n+1)!]^2} \cdot \frac{(n!)^2}{n^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{[(n+1)n!]^2} \cdot \frac{(n!)^2}{n^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{(n+1)^2} \cdot \frac{1}{n^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^n (n+1)}{(n+1)^2 \cdot n^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)} \left(\frac{n+1}{n}\right)^n \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \left(1 + \frac{1}{n}\right)^n \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e}{n+1} \right| = 0 < 1$$

absolutely convergent by R.T

$$(e) \sum_{n=1}^{\infty} \frac{5^n}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{5^n}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{5^n \ln 5}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} (n+1) 5^n \ln 5 = \infty$$

divergent by D.T.

(f) $\sum_{n=1}^{\infty} \frac{n^2+4}{n^{11}+n^7+n+1}$ compare with $\sum_{n=1}^{\infty} \frac{n^2}{n^{11}} = \sum_{n=1}^{\infty} \frac{1}{n^9}$ - $p=9>1$ convergent

$$a_n = \frac{n^2+4}{n^{11}+n^7+n+1}, \quad b_n = \frac{1}{n^9}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^2+4}{n^{11}+n^7+n+1}}{1/n^9} = \lim_{n \rightarrow \infty} \frac{n^9(n^2+4)}{n^{11}+n^7+n+1} = \lim_{n \rightarrow \infty} \frac{n^9 n^2 (1 + \frac{4}{n^2})}{n^{11} (1 + \frac{n^7}{n^{11}} + \frac{n}{n^{11}} + \frac{1}{n^{11}})}$$

$$= \lim_{n \rightarrow \infty} \frac{n^9}{n^{11}} = 1 \neq 0$$

convergent by L.C.T.

$a_n > 0$ for all n , thus \sum is absolutely convergent

13. Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-4)^n$ has the radius of convergence 4. Consider the following pair of series:



(I) $\sum_{n=1}^{\infty} c_n 5^n$ (II) $\sum_{n=1}^{\infty} c_n 3^n$.

Handwritten notes:
 For (I): $x-4=5 \Rightarrow x=9$ (divergent)
 For (II): $x-4=3 \Rightarrow x=7$ (convergent)

interval of convergence is (0, 8)

Which of the following statements is true?

- (a) (I) is convergent, (II) is divergent
- (b) Neither series is convergent
- (c) Both series are convergent
- (d) (I) is divergent, (II) is convergent**
- (e) no conclusion can be drawn about either series.

$$\sum_{n=1}^{\infty} c_n 8^n$$

14. Show that the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges. Then find an upper bound on the error in using s_{10} to approximate the series. (Note that $\ln 2 > 1/2$.)

show that $f(x) = \frac{\ln x}{x^2}$ is continuous, positive and decreasing on $[2, \infty)$

$$\begin{matrix} \ln x > 0 & \text{on } [2, \infty) \\ x^2 > 0 \end{matrix} \Rightarrow f(x) > 0 \text{ on } [2, \infty)$$

$\frac{\ln x}{x^2}$ has discontinuity @ $x=0$, not in $[2, \infty)$
continuous on $[2, \infty)$

$$\left(\frac{\ln x}{x^2}\right)' = \frac{\frac{1}{x} x^2 - 2x \ln x}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} < 0$$

decreasing on $[2, \infty)$.

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{\ln x}{x^2} dx \quad \left| \begin{array}{l} u = \ln x \quad v' = \frac{1}{x^2} \\ u' = \frac{1}{x} \quad v = -\frac{1}{x} \end{array} \right|$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{\ln x}{x} \Big|_2^t - \int_2^t \left(\frac{1}{x}\right) \left(-\frac{1}{x}\right) dx \right]$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{\ln t}{t} + \frac{\ln 2}{2} + \int_2^t \frac{1}{x^2} dx \right) = \lim_{t \rightarrow \infty} \left(-\frac{\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{x} \Big|_2^t \right)$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{t} + \frac{1}{2} \right) = \frac{\ln 2}{2} + \frac{1}{2} - \lim_{t \rightarrow \infty} \frac{1}{t} = \frac{\ln 2 + 1}{2}$$

convergent

$$\boxed{\sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ is convergent by I.T.}}$$

(6) Find an upper bound for R_{10}

$$\int_{10}^{\infty} f(x) dx \leq R_{10} \leq \int_{10}^{\infty} f(x) dx = \frac{1 + \ln 10}{10}$$

$$\boxed{R_{10} \leq \frac{1 + \ln 10}{10}}$$

15. If we represent $\frac{x^2}{4+9x^2}$ as a power series centered at $a = 0$, what is the associated radius of convergence?

$$\frac{1}{4+9x^2} = \frac{1}{4(1+\frac{9}{4}x^2)} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{9}{4}x^2\right)^n \quad \text{geometric, converges if } \left|-\frac{9}{4}x^2\right| < 1$$

$$0 < \frac{9}{4}x^2 < 1$$

$$0 < x^2 < \frac{4}{9}$$

$$-\frac{2}{3} < x < \frac{2}{3} \Rightarrow \boxed{R = \frac{2}{3}}$$

$$\begin{aligned} \frac{x^2}{4+9x^2} &= \frac{1}{4} \cdot x^2 \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{4}\right)^n x^{2n} \\ &= \boxed{\frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{4}\right)^n x^{2n+2}} \end{aligned}$$

16. Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n (3x-1)^n}{n}$.

Ratio Test $a_n = \frac{(-2)^n (3x-1)^n}{n}$, $a_{n+1} = \frac{(-2)^{n+1} (3x-1)^{n+1}}{n+1}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} (3x-1)^{n+1}}{n+1} \cdot \frac{n}{(-2)^n (3x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{-2(3x-1)n}{n+1} \right|$$

$$= 2|3x-1| \lim_{n \rightarrow \infty} \frac{n}{n+1} = 2|3x-1| < 1$$

$$|3x-1| < \frac{1}{2}$$

$$-\frac{1}{2} < 3x-1 < \frac{1}{2}$$

$$\frac{1}{6} < x < \frac{1}{2}$$

$$R = \frac{\frac{1}{2} - \frac{1}{6}}{2} = \frac{1}{6} = R$$

Test end points: $x = 1/6: \sum_{n=1}^{\infty} \frac{(-2)^n (3 \cdot \frac{1}{6} - 1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ - divergent.

$x = 1/2: \sum_{n=1}^{\infty} \frac{(-2)^n (3 \cdot \frac{1}{2} - 1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ - convergent by A.S.T.

$$b_n = \frac{1}{n}, \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

$$n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} < b_n$$

interval of convergence is $\left[\frac{1}{6}, \frac{1}{2} \right]$

17. Which of the following statements is TRUE?

- (a) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- (b) If $a_n > 0$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
- (c) If $\lim_{n \rightarrow \infty} a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
- (d) If $a_n > 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

18. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 3^n}$ converges to s . Use the Alternating Series Theorem to estimate $|s - s_6|$.

alternating.

$$R_6 \leq b_7, \quad b_n = \frac{1}{n^2 3^n},$$

$$\boxed{R_6 \leq \frac{1}{7^2 \cdot 3^7}}$$