

Math 152/172

WEEK in REVIEW 11
Review for the Final

Spring 2017

1. Find the average value of the function $f(x) = \cos(ax + \pi/4)$ on the interval $[0, \frac{\pi}{a}]$, where a is a positive real parameter.

$$\text{fare} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\text{fare} = \frac{1}{\frac{\pi}{a} - 0} \int_0^{\frac{\pi}{a}} \cos(ax + \frac{\pi}{4}) dx \quad \left| \begin{array}{l} u = ax + \frac{\pi}{4} \\ du = a dx \Rightarrow dx = \frac{du}{a} \\ x=0 \Rightarrow u=\frac{\pi}{4} \\ x=\frac{\pi}{a} \Rightarrow u=a\frac{\pi}{a} + \frac{\pi}{4} = \frac{5\pi}{4} \end{array} \right.$$

$$= \frac{1}{\frac{\pi}{a}} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos u \frac{du}{a} = \frac{1}{\pi} \sin u \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{1}{\pi} \left(\sin \frac{5\pi}{4} - \sin \frac{\pi}{4} \right) = \frac{1}{\pi} \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = -\frac{\sqrt{2}}{\pi}$$

$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C,$$

here a and b are constants

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

2. Evaluate the integral $\int_0^1 xe^x dx$

By Parts.

$$\left. \begin{array}{l} u=x \\ u'=1 \\ v=e^x \\ v'=e^x \end{array} \right|$$
$$= xe^x \Big|_0^1 - \int_0^1 e^x dx = e - e^x \Big|_0^1 = e - e + 1 = \boxed{1}$$

$$\int_0^1 xe^{x^2} dx \quad \left. \begin{array}{l} u=x^2 \end{array} \right|$$

Improper.

3. Evaluate the integral $\int_0^1 \frac{4}{(3x+1)(x-1)} dx$

Partial fractions:

$$\frac{4}{(3x+1)(x-1)} = \frac{A}{3x+1} + \frac{B}{x-1} = -\frac{3}{3x+1} + \frac{1}{x-1}$$

$$4 = A(x-1) + B(3x+1)$$

$$x=1: 4 = 4B \Rightarrow B = 1$$

$$x=-\frac{1}{3}: 4 = A(-\frac{1}{3}-1) \Rightarrow A = -3$$

$$= \int_0^1 \left[-\frac{3}{3x+1} + \frac{1}{x-1} \right] dx = - \int_0^1 \frac{3}{3x+1} dx + \int_0^1 \frac{dx}{x-1} \quad p=1 \text{ diverges}$$

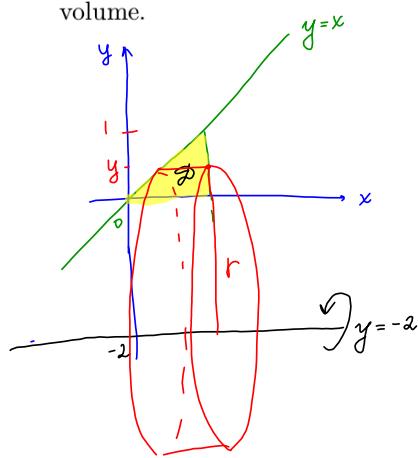
divergent

$\int_1^\infty \frac{dx}{x^p}$

∞ $\int_1^\infty \frac{dx}{x^p}$	converges, if $p > 1$ diverges, if $p \leq 1$
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$\int_0^1 \frac{dx}{x^p}$	converges, if $p < 1$ diverges, if $p \geq 1$
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4. The region $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ is rotated about the horizontal line $y = -2$. Find the generated volume.



Washers.

$$[IR] = 2$$

$$[OR] = x - (-2) = x + 2$$

$$V = \pi \int_0^1 [(x+2)^2 - 2^2] dx = \pi \int_0^1 (x^2 + 4x + 4 - 4) dx \\ = \pi \left[\frac{x^3}{3} + \frac{4x^2}{2} \right]_0^1 = \pi \left(\frac{1}{3} + 2 \right) = \boxed{\frac{7\pi}{3}}$$

shells:

integrate for y , $0 \leq y \leq 1$.

$$r = 2+y, h = [right] - [left] = 1-y$$

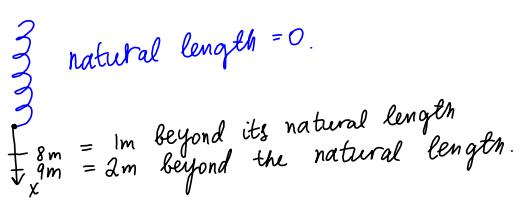
$$V = 2\pi \int_0^1 (2+y)(1-y) dy = \dots = \boxed{\frac{7\pi}{3}}$$

5. Evaluate the integral $\int_{-8}^0 \frac{3x}{\sqrt{x+9}} dx$

$$\left. \begin{array}{l} u = x+9 \Rightarrow x = u-9 \\ du = dx \\ x = -8 \Rightarrow u = -8 + 9 = 1 \\ x = 0 \Rightarrow u = 9 \end{array} \right|$$

$$\begin{aligned}
 &= \int_1^9 \frac{3(u-9) du}{\sqrt{u}} = 3 \int_1^9 (u-9) u^{-\frac{1}{2}} du = 3 \int_1^9 [u \cdot u^{-\frac{1}{2}} - 9u^{-\frac{1}{2}}] du \\
 &= 3 \int_1^9 (u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}) du = 3 \left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 9 \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_1^9 = \dots
 \end{aligned}$$

6. When a spring of natural length 7m is extended to 8m, the force required to hold it in position is 20N. Find the work done (in Joules) when the spring is extended from 8m to 9m.


 natural length = 0.
 \downarrow
 x 8m 9m $= 1\text{m}$ beyond its natural length
 $= 2\text{m}$ beyond the natural length.

$$\left. \begin{array}{l} F = kx \\ 20 = k \cdot 1 \Rightarrow k = 20 \\ F = 20x \\ W = \int_1^2 20x \, dx = \frac{20x^2}{2} \Big|_1^2 = 10(4-1) = \boxed{30 \text{ J}} \end{array} \right\}$$

7. Find the area bounded by the curves $y = 3x$ and $y = x^2 + 2$ from $x = 0$ to $x = 3$.

points of intersection:

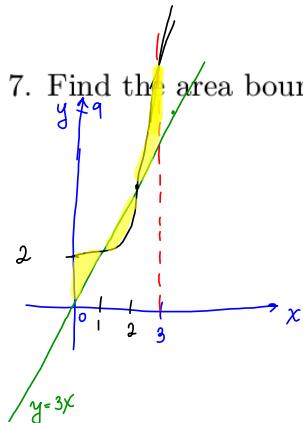
$$3x = x^2 + 2$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0 \Rightarrow x=1, x=2$$

$$A = \int_0^1 (x^2 + 2 - 3x) dx + \int_1^2 [3x - (x^2 + 2)] dx + \int_2^3 (x^2 + 2 - 3x) dx$$

= ...



8. Evaluate the integral $\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx.$

$$\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^4 x \overbrace{\cos^2 x}^{1-\sin^2 x} \cos x \, dx = \int_0^{\pi/2} \sin^4 x (1-\sin^2 x) \cos x \, dx$$

$\left \begin{array}{l} u = \sin x \\ du = \cos x \, dx \\ x=0 \Rightarrow u=\sin 0=0 \\ x=\frac{\pi}{2} \Rightarrow u=\sin \frac{\pi}{2}=1 \end{array} \right.$	$\left \begin{array}{l} \int_0^1 u^4 (1-u^2) \, du = \int_0^1 (u^4 - u^6) \, du \\ = \left[\frac{u^5}{5} - \frac{u^7}{7} \right]_0^1 = \frac{1}{5} - \frac{1}{7} = \boxed{\frac{2}{35}} \end{array} \right.$
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9. Find the integral $\int \frac{1}{\sqrt{4x - x^2 - 3}} dx$

complete the square

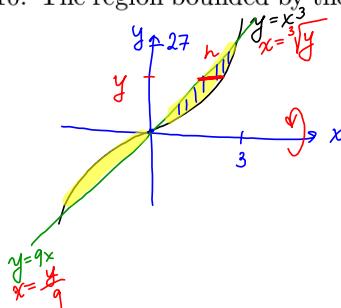
$$\begin{aligned} 4x - x^2 - 3 &= -1(x^2 - 4x + 3) = -1(x^2 - 4x + 2^2 - 2^2 + 3) \\ &= -((x^2 - 4x + 4) - 1) = 1 - (x-2)^2 \end{aligned}$$

$$\int \frac{1}{\sqrt{1-(x-2)^2}} dx \quad \left| \begin{array}{l} x-2 = \sin t \Rightarrow t = \arcsin(x-2) \\ dx = \cos t dt \\ \sqrt{1-(x-2)^2} = \sqrt{1-\sin^2 t} \\ = \sqrt{\cos^2 t} = \cos t \end{array} \right. = \int \frac{\cos t dt}{\cos t} = \int dt = t + C = \boxed{\arcsin(x-2) + C}$$

$$u = x-2$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

10. The region bounded by the curves $y = x^3$ and $y = 9x$ is rotated about the x -axis. Find the volume generated.



points of intersection:

$$x^3 = 9x$$

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x(x-3)(x+3) = 0 \Rightarrow x=0, x=3, x=-3.$$

washers:

$$[IR] = x^3$$

$$[OR] = 9x$$

$$V = 2\pi \int_0^3 [(9x)^2 - (x^3)^2] dx = 2\pi \int_0^3 (81x^2 - x^6) dx = \dots$$

shells:

integrate for y .

$$0 \leq y \leq 27 \Rightarrow 0 \leq y \leq x^3$$

$$r = y$$

$$h = \sqrt[3]{y} - \frac{y}{9}$$

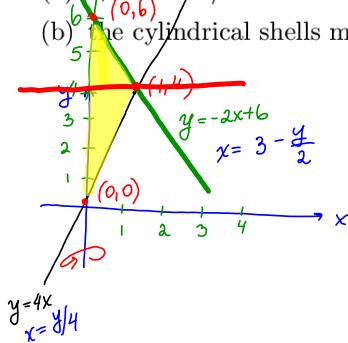
$$V = 4\pi \int_0^{27} y \left(\sqrt[3]{y} - \frac{y}{9} \right) dy = 4\pi \int_0^{27} \left(y^{4/3} - \frac{y^2}{9} \right) dy = \dots$$

$$V = 2\pi \int_a^b [radius][height] dx$$

11. The region bounded by the lines $x=0$, $y = -2x + 6$ and $y = 4x$ is rotated about the y -axis. Set up, but don't evaluate, integrals which give the volume generated using

(a) the washer/disk method

(b) the cylindrical shells method.



points of intersection:

$$-2x+6=4x \Rightarrow x=1, y=4$$

$$x=0 \Rightarrow y=6$$

washers: integrate for y , $0 \leq y \leq 6$

$$[IR] = 0$$

outer radius: $[OR] = \frac{y}{4}$, when $0 \leq y \leq 4$

$[OR] = 3 - \frac{y}{2}$, when $4 \leq y \leq 6$.

$$V = \pi \int_0^4 \left(\frac{y}{4}\right)^2 dy + \pi \int_4^6 \left(3 - \frac{y}{2}\right)^2 dy$$

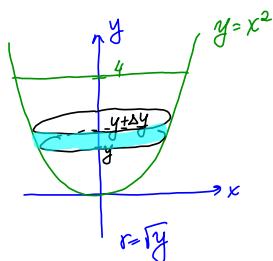
shells: integrate for x , $0 \leq x \leq 1$

$$[\text{radius}] = x$$

$$[\text{height}] = [\text{top}] - [\text{bottom}] = -2x+6 - 4x \\ = 6 - 6x$$

$$V = 2\pi \int_0^1 x(6 - 6x) dx$$

12. A tank was constructed by rotating about the y -axis the part of the parabola $y = x^2$ such that depth of the tank is 4 ft. The tank is then filled with a liquid solution weighing 60 lb/ft^3 . Find the work done in pumping out the tank.



$$W = \cancel{\int_0^4} \cancel{A(y) \cancel{\text{dist}(y)} dy}$$

distance to the top
of a cross-sectional area

Take a "slice" of the liquid between y and $y + \Delta y$.

$$\begin{aligned} [\text{weight}] &= \cancel{\int_0^4} [\text{volume}] \\ &= \cancel{\int_0^4} \pi r^2 \Delta y = \cancel{\int_0^4} \pi (4-y)^2 \Delta y = \pi(60) y \Delta y \end{aligned}$$

$$[\text{distance}] = 4 - y$$

$$W = 60\pi \int_0^4 y (4-y) dy = \dots$$

13. Which of these integrals represents the length of the curve $y = x^4$ from $x = 0$ to $x = 1$?

(a) $\int_0^1 \sqrt{1+x^8} dx$

$$L = \int_0^1 \sqrt{1 + [y'(x)]^2} dx$$
$$y'(x) = 4x^3$$

(b) $\int_0^1 \sqrt{1+4x^3} dx$

$$= \int_0^1 \sqrt{1 + (4x^3)^2} dx = \int_0^1 \sqrt{1 + 16x^6} dx$$

(c) $\int_0^1 \sqrt{1+16x^6} dx$

(d) $2\pi \int_0^1 x^4 \sqrt{1+16x^6} dx$

(e) $\int_0^1 \sqrt{1+x^4} dx$

14. By comparing the functions $\frac{1}{1+x^5}$ and $\frac{1}{x^5}$ what conclusion can be drawn about $\int_1^\infty \frac{1}{1+x^5} dx$?

- (a) Its value is 1.
- (b) Its value is $1/2$.

$$\frac{1}{1+x^5} < \frac{1}{x^5}$$

$$\int_1^\infty \frac{dx}{x^5} - \text{convergent}, p = 5 > 1$$

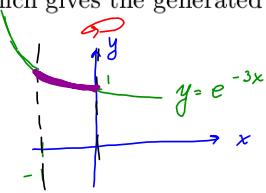
By Comparison Thm for improper integrals, $\int_1^\infty \frac{dx}{1+x^5}$ converges.

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- (c) It diverges.
 - (d) It converges.
 - (e) No conclusion is possible.

15. Does the integral $\int_0^1 \frac{1+x}{\sqrt{x}} dx$ diverge? NO

$$\begin{aligned}
 \int_0^1 \frac{1+x}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \int_t^1 (1+x) x^{-1/2} dx = \lim_{t \rightarrow 0^+} \int_t^1 (x^{-1/2} + x^{1/2}) dx \\
 &= \lim_{t \rightarrow 0^+} \left[\frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} \right]_t^1 = 2 + \frac{2}{3} - \lim_{t \rightarrow 0^+} \left(2t^{1/2} + \frac{2}{3} t^{3/2} \right) \\
 &= 2 + \frac{2}{3}
 \end{aligned}$$

16. The curve $y = e^{-3x}$ from $x = -1$ to $x = 0$ is rotated about the y -axis. Set up, but don't evaluate, integral which gives the generated surface area.



$$\begin{aligned} S.A. &= 2\pi \int_{-1}^0 x \sqrt{1 + [y'(x)]^2} dx \\ y'(x) &= -3e^{-3x} \\ S.A. &= 2\pi \int_{-1}^0 x \sqrt{1 + (-3e^{-3x})^2} dx \\ &= 2\pi \int_{-1}^0 x \sqrt{1 + 9e^{-6x}} dx \end{aligned}$$

17. Given a positive series with general term a_n .

- (a) TRUE **FALSE** If $\lim_{n \rightarrow \infty} a_n = 0$ then the series converges.
- (b) TRUE **FALSE** If $a_n \geq \frac{1}{n^4}$ then the series converges.
- (c) TRUE **FALSE** If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ then the series converges. R.T. fails.
- (d) TRUE **FALSE** If $a_n \leq \frac{1}{\sqrt{n}}$ then the series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent, } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} - p=4>1 \text{ convergent.}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad p=\frac{1}{2}<1 \text{ divergent}$$

18. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x-1)^n}{\sqrt[5]{n^2}}$

$$a_n = \frac{(-1)^{n+1}(2x-1)^n}{n^{2/5}}, \quad a_{n+1} = \frac{(-1)^{n+2}(2x-1)^{n+1}}{(n+1)^{2/5}}$$

$$\text{R.T. } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}(2x-1)^{n+1}}{(n+1)^{2/5}}}{\frac{(-1)^{n+1}(2x-1)^n}{n^{2/5}}} \right| = |2x-1| < 1$$

$$\begin{aligned} -1 &< 2x-1 < 1 \\ 0 &< 2x < 2 \\ 0 &< x < 1 \end{aligned} \Rightarrow R = \frac{1-0}{2} = \frac{1}{2}$$

check end-points.

$$x=0: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{\sqrt[5]{n^2}} = \sum_{n=1}^{\infty} \frac{(-1)}{\sqrt[5]{n^2}} = -\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^2}} - p\text{-series}$$

$p=2/5 < 1$
divergent.

$$x=1: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2-1)^n}{\sqrt[5]{n^2}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2/5}} - \text{alternating}, \quad b_n = \frac{1}{n^{2/5}}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n^{2/5}} = 0$$

$$(n+1)^{2/5} \geq n^{2/5}, \text{ if } n \geq 1$$

$$\frac{1}{(n+1)^{2/5}} \leq \frac{1}{n^{2/5}} \Rightarrow b_{n+1} \leq b_n$$

convergent by A.S.T.

$(0, 1]$

19. Compute $\lim_{x \rightarrow 0} \frac{x^4 - \sin(x^4)}{1 - \cos(x^6)}$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\begin{aligned}\sin(x^4) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = x^4 - \frac{(x^4)^3}{3!} + \frac{(x^4)^5}{5!} - \dots \\ &= x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \dots\end{aligned}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\begin{aligned}\cos(x^6) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^6)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{12n}}{(2n)!} \\ &= 1 - \frac{x^{12}}{2!} + \frac{x^{24}}{4!} - \dots\end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{x^4 - [x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{5!} - \dots]}{1 - [1 - \frac{x^{12}}{2} + \frac{x^{24}}{24} - \dots]} = \lim_{x \rightarrow 0} \frac{\frac{x^{12}}{6} - \frac{x^{20}}{5!} + \dots}{\frac{x^{12}}{2} - \frac{x^{24}}{24} + \dots}$$

$$= \lim_{x \rightarrow 0} \frac{x^{12} \cdot \left[\frac{1}{6} - \frac{x^8}{5!} + \dots \right]}{x^{12} \left[\frac{1}{2} - \frac{x^{12}}{24} + \dots \right]} = \frac{1/6}{1/2} = \boxed{\frac{1}{3}}$$

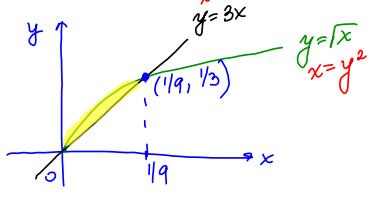
20. The series $\sum_{n=2017}^{\infty} \frac{(n!)^6}{((3n)!)^6} = \sum_{n=2017}^{\infty} \left(\frac{n!}{(3n)!} \right)^6$

- (a) Diverges by the Integral Test
- (b) Diverges by the Comparison Test
- (c) Diverges by the Ratio Test
- (d) Converges by the Ratio Test
- (e) Diverges because $\lim_{n \rightarrow \infty} a_n = 0$.

R.T. $a_n = \left[\frac{n!}{(3n)!} \right]^6$
 $a_{n+1} = \left[\frac{(n+1)!}{3(n+1)!} \right]^6 = \left[\frac{(n+1)n!}{(3n+3)!} \right]^6$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{\frac{(n+1)n!}{(3n+3)!}}{\frac{n!}{(3n)!}} \right]^6 = \lim_{n \rightarrow \infty} \left[\frac{(n+1)(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!} \right]^6 = 0 < 1$$

21. Find the area bounded by the curves $y = 3x$ and $y = \sqrt{x}$.



Points of intersection:

$$3x = \sqrt{x}$$

$$9x^2 = x$$

$$9x^2 - x = 0$$

$$x(9x - 1) = 0$$

$$x_1 = 0, \quad x_2 = 1/9$$

$$A = \int_0^{1/9} (\sqrt{x} - 3x) dx = \int_0^{1/3} \left(\frac{y}{3} - y^2\right) dy$$

22. A trigonometric substitution converts the integral $\int \sqrt{x^2 + 20x + 75} dx$ to

- (a) $5 \int \tan^3 \theta d\theta$
- (b)** $25 \int \tan^2 \theta \sec \theta d\theta$
- (c) $25 \int \sin^3 \theta d\theta$
- (d) $5 \int \sin^2 \theta \cos \theta d\theta$
- (e) $5 \int \tan \theta \sec^2 \theta d\theta$

$$\begin{aligned}
 & \frac{x^2 + 20x + 75}{2 \cdot 10} = \frac{x^2 + 20x + 10^2 - 10^2 + 75}{2 \cdot 10} \\
 & = \frac{(x+10)^2 - 25}{2 \cdot 10} \\
 & \int \sqrt{(x+10)^2 - 25} dx \quad \left| \begin{array}{l} x+10 = 5 \sec t \\ dx = 5 \sec t \tan t dt \\ \sqrt{(x+10)^2 - 25} = \sqrt{25 \sec^2 t - 25} = 5 \tan t \end{array} \right. \\
 & = \int 5 \tan t \cdot 5 \sec t \tan t dt
 \end{aligned}$$

23. Find the average value of the function $f(x) = \sin^3 x$ on the interval $[0, \frac{\pi}{2}]$.

$$\begin{aligned}
 \text{average} &= \frac{1}{\frac{\pi}{2}-0} \int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^2 t \sin t \, dt = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (1-\cos^2 t) \sin t \, dt \\
 &= -\frac{2}{\pi} \int_1^0 (1-u^2) \, du = \frac{2}{\pi} \int_0^1 (1-u^2) \, du \\
 &= \frac{2}{\pi} \left(u - \frac{u^3}{3} \right) \Big|_0^1 = \frac{2}{\pi} \cdot \frac{2}{3} = \boxed{\frac{4}{3\pi}}
 \end{aligned}$$

$\begin{cases} u = \cos t \\ du = -\sin t \, dt \\ t=0 \Rightarrow u=1 \\ t=\frac{\pi}{2} \Rightarrow u=0 \end{cases}$

24. Evaluate the integral $\int x^3 \sin 3x \, dx = \boxed{-\frac{x^3}{3} \cos 3x + \frac{3x^2}{9} \sin 3x + \frac{6x}{27} \cos 3x - \frac{6}{81} \sin 3x + C}$

Tabulation	
u	v'
+ x^3	$\sin 3x$
- $3x^2$	$-\frac{1}{3} \cos 3x$
+ $6x$	$-\frac{1}{9} \sin 3x$
- 6	$\frac{1}{27} \cos 3x$
+ 0	$\frac{1}{81} \sin 3x$

26. Set up, but don't evaluate, integral which gives the arc length of the curve

$$x = 2017 + \cos(2t), \quad y = t - \sin(2t), \quad 0 \leq t \leq \pi/2.$$

Circle the correct answer:

(a) $\int_0^{\pi/2} \sqrt{2018 + t^2 + 2\cos(2t) - 2\sin(2t)} dt$

(b) $\int_0^{\pi/2} \sqrt{2017 + t^2 + 2\cos(2t) - 2\sin(2t)} dt$

(c) $\int_0^{\pi/2} \sqrt{2017 - 4\cos(2t) + 4\sin(2t)} dt$

(d) $\int_0^{\pi/2} \sqrt{5 - 4\cos(2t)} dt$

(e) $\int_0^{\pi/2} \sqrt{6 - 4\cos(2t)} dt$

$$L = \int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

$$x'(t) = -2\sin 2t$$

$$y'(t) = 1 - 2\cos 2t$$

$$[x'(t)]^2 + [y'(t)]^2 = 4\sin^2 2t + (1 - 2\cos 2t)^2$$

$$= 4\sin^2 2t + 1 - 4\cos 2t + 4\cos^2 2t$$

$$= 4(\sin^2 2t + \cos^2 2t) + 1 - 4\cos 2t$$

$$= 5 - 4\cos 2t$$

$$L = \int_0^{\pi/2} \sqrt{5 - 4\cos 2t} dt$$

27. Determine whether the integral $\int_1^\infty \frac{1}{(x-3)^4} dx$ is divergent or convergent.

discontinuous @ x=3

$$\int_1^\infty \frac{dx}{(x-3)^4} = \underbrace{\int_1^3 \frac{dx}{(x-3)^4}}_{p=4>1} + \int_3^\infty \frac{dx}{(x-3)^4}$$

divergent

28. Find $\int \frac{x^2 + 1}{x^3 + 2x^2 + x} dx$

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x+1)^2$$

partial fractions:

$$\frac{x^2 + 1}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 + 1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$x^2 + 1 = Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$x^2: 1 = A + B \Rightarrow \boxed{B=0}$$

$$x: 0 = 2A + B + C \Rightarrow \boxed{C=1}$$

$$1: \boxed{1 = A}$$

$$\begin{aligned} \int \frac{(x^2 + 1) dx}{x^3 + 2x^2 + x} &= \left[\frac{1}{x} + \frac{2}{(x+1)^2} \right] dx \\ &= \boxed{\ln|x| - \frac{2}{x+1} + C} \end{aligned}$$

29. Describe the surface having the equation $x^2 + y^2 + z^2 - 10x + 2z + 1 = 0$.

Complete the square:

$$x^2 - 10x + y^2 + z^2 + 2z - 1 = 0$$

$$(x^2 - 10x + 25) + y^2 + (z^2 + 2z + 1) - 25 - 1 + 1 = 0$$

$$(x-5)^2 + y^2 + (z+1)^2 = 25$$

Sphere with center $(5, 0, -1)$
radius 5.

30. Compute $\int_{-1}^1 \frac{1}{x^6} dx$ divergent

discontinuous @ 0

$$\int_{-1}^1 \frac{dx}{x^6} = \int_{-1}^0 \frac{dx}{x^6} + \int_0^1 \frac{dx}{x^6}$$

$p=6 > 1$ - divergent

31. Which of the following series are convergent?

$$(a) \sum_{n=1}^{\infty} \frac{2017^n}{n!}$$

$$(b) \sum_{n=1}^{\infty} \frac{2017^n}{n + 2018^n}$$

(a) ~~do~~ R.T., $a_n = \frac{2017^n}{n!}$, $a_{n+1} = \frac{2017^{n+1}}{(n+1)!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2017^{n+1}}{(n+1)!}}{\frac{2017^n}{n!}} \right| = \lim_{n \rightarrow \infty} \frac{2017}{n+1} = 0 < 1$$

convergent

(b) $\frac{2017^n}{n+2018^n} < \frac{2017^n}{2018^n} = \left(\frac{2017}{2018}\right)^n$, $\frac{2017}{2018} < 1$

$\sum_{n=1}^{\infty} \left(\frac{2017}{2018}\right)^n$ is geometric,
 $r = \frac{2017}{2018} < 1$,
 convergent.

$\sum_{n=1}^{\infty} \frac{2017^n}{n+2018^n}$ converges by C.T.

32. Compute $\sum_{n=0}^{\infty} \frac{2017^{n-1}}{2016^n}$. $= \sum_{n=0}^{\infty} \frac{1}{2017} \cdot \frac{2017^n}{2016^n} = \sum_{n=0}^{\infty} \frac{1}{2017} \left(\frac{2017}{2016}\right)^n$

geometric, $r = \frac{2017}{2016} > 1$

divergent

33. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[4]{k}}$ is

- (a) divergent to ∞
- (b) divergent to $-\infty$
- (c) divergent but not to $\pm\infty$

~~(d)~~ absolutely convergent
~~(e)~~ conditionally convergent

3

~~(e)~~ conditionally convergent

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{\sqrt[4]{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}} - \begin{matrix} p\text{-series} \\ p=1/4 < 1 \\ \text{divergent} \end{matrix}$$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[4]{k}}$ - alternating, $b_k = \frac{1}{\sqrt[4]{k}}$

$$\lim_{k \rightarrow \infty} b_k = \lim_{k \rightarrow \infty} \frac{1}{\sqrt[4]{k}} = 0$$

since $\sqrt[4]{k+1} > \sqrt[4]{k}, k \geq 1$

$$\text{then } \frac{1}{\sqrt[4]{k+1}} < \frac{1}{\sqrt[4]{k}} \Rightarrow b_{k+1} < b_k$$

convergent by A.S.T.

(v) orthogonality condition

34. Find the value(s) of x such that the vectors $\langle x, -1, 1 \rangle$ and $\langle 1, -x^2, x^3 \rangle$ are orthogonal.

The dot product = 0.

$$\langle x, -1, 1 \rangle \cdot \langle 1, -x^2, x^3 \rangle = x + x^2 + x^3 = 0$$
$$x(x^2 + x + 1) = 0$$
$$\boxed{x=0} \text{ or } x^2 + x + 1 = 0$$
$$\Delta = 1 - 4 = -3 < 0$$

no solutions.

35. Find the Taylor series for $f(x) = x^3 + x^2 + 3$ about $x = 3$.

$f(x) = x^3 + x^2 + 3$	$f(3) = 3^3 + 3^2 + 3 = 39$
$f'(x) = 3x^2 + 2x$	$f'(3) = 3 \cdot 9 + 2 \cdot 3 = 33$
$f''(x) = 6x + 2$	$f''(3) = 6(3) + 2 = 20$
$f'''(x) = 6$	$f'''(3) = 6$
$f^{(4)}(x) = 0$	$f^{(4)}(3) = 0$

$$\begin{aligned}
 x^3 + x^2 + 3 &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{6}(x-3)^3 \\
 &= 39 + 33(x-3) + \frac{20}{2}(x-3)^2 + \frac{6}{6}(x-3)^3 \\
 &= \boxed{39 + 33(x-3) + 10(x-3)^2 + (x-3)^3}
 \end{aligned}$$

36. Find a power series centered at $x = 0$ for the function $f(x) = \frac{x}{1-8x^3}$, and determine the radius of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1-8x^3} = \sum_{n=0}^{\infty} (8x^3)^n = \sum_{n=0}^{\infty} 8^n x^{3n}, \quad |8x^3| < 1$$

$$\frac{x}{1-8x^3} = x \sum_{n=0}^{\infty} 8^n x^{3n} = \boxed{\sum_{n=0}^{\infty} 8^n x^{3n+1}}$$

$$|8x^3| < 1$$

$$|x^3| < \frac{1}{8}$$

$$|x| < \frac{1}{2} \Rightarrow \boxed{R = \frac{1}{2}}$$

37. Find the angle between the vectors $\langle 1, 2, 1 \rangle$ and $\langle 3, 3, 0 \rangle$.

$$\begin{aligned}\cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{\langle 1, 2, 1 \rangle \cdot \langle 3, 3, 0 \rangle}{\sqrt{1+4+1} \cdot \sqrt{9+9}} = \frac{3+6}{\sqrt{6} \sqrt{18}} = \frac{9}{\sqrt{3} \cdot 6} \\ &= \frac{\sqrt{3}}{2}, \quad \theta = \frac{\pi}{6}\end{aligned}$$

38. Evaluate the integral $\int_0^{1/3} \frac{1}{1+x^7} dx$ as infinite series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

$$\int_0^{1/3} \frac{dx}{1+x^7} = \sum_{n=0}^{\infty} (-1)^n \left[\int_0^{1/3} x^{7n} dx \right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \Big|_0^{1/3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{3}\right)^{7n+1}}{7n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(7n+1) 3^{7n+1}}}$$

39. Find the third degree Taylor polynomial for $f(x) = \ln x$ about $x = 5$.

$$T_3(x) = f(5) + f'(5)(x-5) + \frac{f''(5)}{2}(x-5)^2 + \frac{f'''(5)}{6}(x-5)^3$$

$f(x) = \ln x$	$f(5) = \ln 5$
$f'(x) = \frac{1}{x}$	$f'(5) = \frac{1}{5}$
$f''(x) = -\frac{1}{x^2}$	$f''(5) = -\frac{1}{25}$
$f'''(x) = \frac{2}{x^3}$	$f'''(5) = \frac{2}{125}$

$$\begin{aligned} T_3(x) &= \ln 5 + \frac{1}{5}(x-5) - \frac{1}{25 \cdot 2}(x-5)^2 + \frac{2}{125 \cdot 6}(x-5)^3 \\ &= \boxed{\ln 5 + \frac{1}{5}(x-5) - \frac{1}{50}(x-5)^2 + \frac{1}{375}(x-5)^3} \end{aligned}$$

40. Consider the points $A(0, 1, 4)$, $B(2, 1, 3)$, and $C(1, -1, 0)$.

- (a) Find a unit vector orthogonal to the plane determined by the given points.
- (b) Find the area of the triangle with vertices A, B, and C.

a vector \vec{c} orthogonal to the plane is
 $\vec{c} = \vec{AB} \times \vec{AC}$

$$\vec{AB} = \langle 2-0, 1-1, 3-4 \rangle = \langle 2, 0, -1 \rangle$$

$$\vec{AC} = \langle 1-0, -1-1, 0-4 \rangle = \langle 1, -2, -4 \rangle$$

$$\begin{aligned}\vec{c} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 1 & -2 & -4 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & -1 \\ -2 & -4 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} \\ &= \vec{i}(0-2) - \vec{j}(2+1) + \vec{k}(-4-0) \\ &= -2\vec{i} + 7\vec{j} - 4\vec{k}\end{aligned}$$

$$\vec{u} = \frac{\vec{c}}{|\vec{c}|} = \frac{\langle -2, 7, -4 \rangle}{\sqrt{4+49+16}} = \boxed{\frac{\langle -2, 7, -4 \rangle}{\sqrt{69}}}$$

$$(b) A = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \boxed{\frac{\sqrt{169}}{2}}$$

41. Which of the following statements is most accurately describes the convergence or divergence of the improper integral $\int_1^\infty \frac{x}{\sqrt{x^7 + 77}} dx$?

- (a) The integral converges because $\frac{x}{\sqrt{x^7 + 77}} < \frac{1}{x^{7/2}}$ and the integral $\int_1^\infty \frac{1}{x^{7/2}} dx$ converges.
- (b) The integral converges because $\frac{x}{\sqrt{x^7 + 77}} < \frac{1}{x^6}$ and the integral $\int_1^\infty \frac{1}{x^6} dx$ converges.
- (c) The integral converges because $\frac{x}{\sqrt{x^7 + 77}} < \frac{1}{x^{5/2}}$ and the integral $\int_1^\infty \frac{1}{x^{5/2}} dx$ converges.
- (d) The integral diverges because $\frac{x}{\sqrt{x^7 + 77}} \geq \frac{1}{x^{5/2}}$ and the integral $\int_1^\infty \frac{1}{x^{5/2}} dx = \infty$.
- (e) The integral diverges because $\frac{x}{\sqrt{x^7 + 77}} \geq \frac{1}{x^6}$ and the integral $\int_1^\infty \frac{1}{x^6} dx = \infty$.

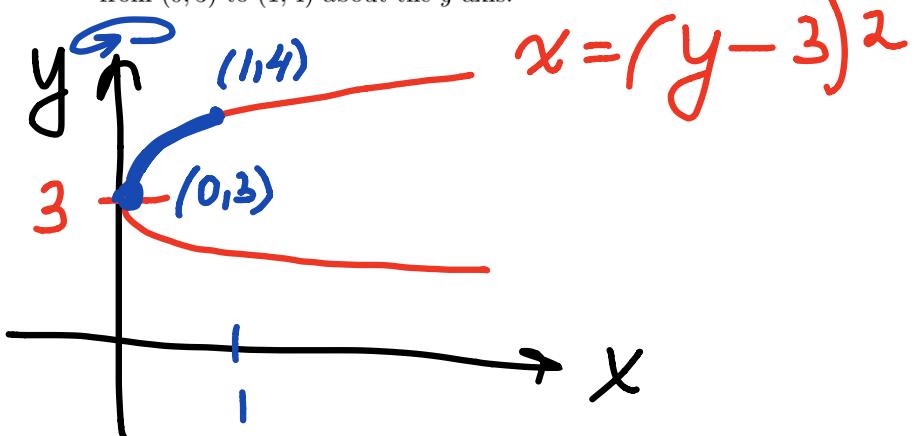
$$\sqrt{x^7 + 77} \geq \sqrt{x^7} = x^{7/2}$$

$$\frac{x}{\sqrt{x^7 + 77}} \leq \frac{x}{x^{7/2}} = \frac{1}{x^{5/2}}$$

$$\int_1^\infty \frac{dx}{x^{5/2}}$$

$p = 5/2 > 1$
convergent.

42. Set up the integral that will compute the area of the surface obtained by revolving the curve $x = (y - 3)^2$ from $(0, 3)$ to $(1, 4)$ about the y -axis.



$$\text{S.A.} = 2\pi \int_{3}^{4} x(y) \sqrt{1 + [x'(y)]^2} dy$$

$$x'(y) = 2(y-3)$$

$$= \boxed{2\pi \int_{3}^{4} (y-3)^2 \sqrt{1 + 4(y-3)^2} dy}$$