$$\boxed{\int_a^b f(x)\,dx = F(b)-F(a), \text{ where } F'(x) = f(x)}$$

Math 152/172 **WEEK in REVIEW 1** **Spring 2017**

1. Evaluate the definite integral.

(a) $\displaystyle\int_1^2 \frac{x^2+1}{\sqrt{x}}dx = \int_1^2 (x^2+1)x^{-1/2}dx = \int_1^2 (\overbrace{x^2\cdot x^{-1/2}}^{x^{2-1/2}=x^{3/2}} + x^{-1/2})\,dx$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}+C \quad n\neq -1}$$

$$= \int_1^2 (x^{3/2}+x^{-1/2})\,dx = \left[\frac{x^{3/2+1}}{3/2+1} + \frac{x^{-1/2+1}}{-1/2+1}\right]_1^2$$

$$= \left[\frac{2}{5}x^{5/2} + 2x^{1/2}\right]_1^2 = \frac{2}{5}2^{5/2} + 2\cdot 2^{1/2} - \frac{2}{5} - 2$$

$$= \frac{2}{5}\cdot 4\sqrt{2} + 2\sqrt{2} - \frac{12}{5} = \frac{8\sqrt{2}}{5} + 2\sqrt{2} - \frac{12}{5} = \boxed{\frac{18\sqrt{2}}{5} - \frac{12}{5}}$$

(b) $\displaystyle\int_1^8 \sqrt[3]{x}(x^2-\sqrt{x})dx = \int_1^8 x^{1/3}(x^2 - x^{1/2})\,dx = \int_1^8 (\overbrace{x^{1/3}x^2}^{x^{2+\frac{1}{3}}=x^{\frac{7}{3}}} - \underbrace{x^{1/3}\cdot x^{1/2}}_{x^{\frac{1}{3}+\frac{1}{2}}=x^{\frac{5}{6}}})\,dx$

$$= \int_1^8 (x^{7/3} - x^{5/6})\,dx = \left[\frac{x^{\frac{7}{3}+1}}{\frac{7}{3}+1} - \frac{x^{\frac{5}{6}+1}}{\frac{5}{6}+1}\right]_1^8$$

$$= \left[\frac{3}{10}x^{10/3} - \frac{6}{11}\cdot x^{11/6}\right]_1^8 = \frac{3}{10}8^{10/3} - \frac{6}{11}8^{11/6} - \frac{3}{10} + \frac{6}{11}$$

$$= \frac{3}{10}\cdot 8^3\cdot 8^{1/3} - \frac{6}{11}\cdot 8\cdot 8^{5/6} - \frac{33-60}{110}$$

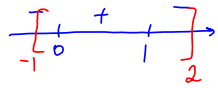
$$= \boxed{\frac{3072}{10} - \frac{48}{11}8^{5/6} + \frac{27}{110}}$$

(c) $\displaystyle\int_{-1}^2 |x-x^2|dx$

$$|x-x^2| = \begin{cases} x-x^2, & \text{if } x-x^2\geq 0 \\ -(x-x^2), & \text{if } x-x^2<0 \end{cases}$$

$x-x^2\geq 0$

$x(1-x)\geq 0$



$x=2$: $2(1-2)<0$

$x=1/2$: $1/2(1-1/2)>0$

$x=-1$: $-1(1+1)<0$

$$\boxed{\begin{array}{l} x-x^2\geq 0 \text{ on } [0,1] \\ x-x^2<0 \text{ on } (-\infty,0)\cup(1,\infty) \end{array}}$$

$$|x-x^2| = \begin{cases} x-x^2, & \text{if } 0\leq x\leq 1 \\ -(x-x^2), & \text{if } (x<0) \text{ or } (x>1) \end{cases}$$

$$= \int_{-1}^0 [-(x-x^2)]\,dx + \int_0^1 (x-x^2)\,dx + \int_1^2 [-(x-x^2)]\,dx$$

$$= \left[-\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= 0 - \left(-\frac{(-1)^2}{2} + \frac{(-1)^3}{3}\right) + \frac{1}{2} - \frac{1}{3} - 0 - \left(\frac{2^2}{2} - \frac{2^3}{3} - \frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{2} + \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{3}} - 2 + \frac{8}{3} + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{3}{2} - 2 + \frac{7}{3} = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \boxed{\frac{11}{6}}$$

(d) $\displaystyle\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left[(\sqrt{x})^2 + 2\sqrt{x}\cdot\frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2\right] dx$

$= \displaystyle\int_4^9 \left(x+2+\frac{1}{x}\right)dx = \left[\frac{x^2}{2}+2x+\ln|x|\right]_4^9 = \frac{81}{2}+18+\ln 9-\frac{16}{2}-8-\ln 4$

$= \dfrac{65}{2}+10+\ln 9-\ln 4 = \boxed{\dfrac{85}{2}+\ln\dfrac{9}{4}}$

(e) $\displaystyle\int_0^1 \left(x^2+1+\frac{1}{x^2+1}\right)dx = \left[\frac{x^3}{3}+x+\arctan x\right]_0^1 = \frac{1}{3}+1+\overset{\pi/4}{\cancel{\arctan 1}}-\overset{0}{\cancel{\arctan 0}}$

$= \boxed{\dfrac{4}{3}+\dfrac{\pi}{4}}$

(f) $\displaystyle\int_{\pi/2}^{\pi} \sec x \tan x \, dx = \sec x\Big]_{\pi/2}^{\pi} = \sec\pi - \sec\pi/2 = -1-\frac{1}{0} = -\infty$

2. If $f(3)=12$, f' is continuous, and $\displaystyle\int_3^5 f'(x)\,dx = 20$, what is the value of $f(5)$?

$\underbrace{\displaystyle\int_3^5 f'(x)\,dx}_{20} = f(5)-\underbrace{f(3)}_{12}$ (by the Fundamental Theorem of Calculus)

$20 = f(5)-12 \Rightarrow f(5) = 20+12 = \boxed{32}$

$$\int f(g(x))g'(x)\,dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right| = \int f(u)\,du$$

$$\int_a^b f(g(x))g'(x)\,dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \\ a \to g(a) \\ b \to g(b) \end{array} \right| = \int_{g(a)}^{g(b)} f(u)\,du$$

3. Find the integral.

$\boxed{\int e^x dx = e^x + C}$

(a) $\displaystyle\int e^{2016x}\,dx$ $\quad \left| \begin{array}{l} u = 2016x \\ du = 2016\,dx \Rightarrow dx = \frac{du}{2016} \end{array} \right|$

$$= \int e^u \frac{du}{2016} = \frac{1}{2016}\int e^u\,du = \frac{1}{2016}e^u + C$$

$$= \boxed{\frac{1}{2016}e^{2016x} + C}$$

$\boxed{\int \sin x\,dx = -\cos x + C}$

(b) $\displaystyle\int_2^4 \sin(4\pi x)\,dx$ $\quad \left| \begin{array}{l} u = 4\pi x \\ du = 4\pi\,dx \Rightarrow dx = \frac{du}{4\pi} \\ x = 2 \Rightarrow u = 4\pi(2) = 8\pi \\ x = 4 \Rightarrow u = 4\pi(4) = 16\pi \end{array} \right|$ $\displaystyle = \int_{8\pi}^{16\pi} \sin u \frac{du}{4\pi} = \frac{1}{4\pi}\int_{8\pi}^{16\pi} \sin u\,du$

$$= \frac{1}{4\pi}\left(-\cos u\right)\Big|_{8\pi}^{16\pi} = \frac{1}{4\pi}\left(-\overset{1}{\cancel{\cos 16\pi}} + \overset{1}{\cancel{\cos 8\pi}}\right)$$

$$= \frac{1}{4\pi}(-1+1) = \boxed{0}$$

(c) $\displaystyle\int_0^{\pi/2} \cos^7 x \overbrace{\sin(2x)}^{2\sin x\cos x}\,dx$ $\displaystyle = \int_0^{\pi/2} \cos^7 x\,(2\sin x\cos x)\,dx = 2\int_0^{\pi/2} \cos^8 x \sin x\,dx$ $\quad \left| \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ x = 0 \Rightarrow u = \cos 0 = 1 \\ x = \frac{\pi}{2} \Rightarrow u = \cos\frac{\pi}{2} = 0 \end{array} \right|$

$$= 2\int_1^0 u^8(-du) = 2\int_0^1 u^8\,du = 2\cdot\frac{u^9}{9}\Big]_0^1 = \boxed{\frac{2}{9}}$$

(d) $\displaystyle\int_0^1 x^4 e^{9x^5-8}\,dx$ $\quad \left| \begin{array}{l} u = 9x^5 - 8 \\ du = 45x^4\,dx \Rightarrow x^4\,dx = \frac{du}{45} \\ x = 0 \Rightarrow u = 9\cdot 0 - 8 = -8 \\ x = 1 \to u = 9 - 8 = 1 \end{array} \right|$ $\displaystyle = \int_{-8}^{1} e^u \cdot \frac{du}{45} = \frac{1}{45}\int_{-8}^{1} e^u\,du$

$$= \frac{1}{45}e^u\Big]_{-8}^{1} = \boxed{\frac{1}{45}\left(e - e^{-8}\right)}$$

(e) $\displaystyle\int \frac{\overbrace{15x^2+50x}^{5(3x^2+10x)}}{x^3+5x^2+8}dx$ $\left| \begin{array}{l} u = x^3+5x^2+8 \\ du = (3x^2+10x)dx \end{array} \right| = \displaystyle\int \frac{5du}{u} = 5\int \frac{du}{u}$

$= 5\ln|u| + C = \boxed{5\ln|x^3+5x^2+8| + C}$

(f) $\displaystyle\int \frac{dx}{x\sqrt[3]{\ln x}}$ $\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right| = \displaystyle\int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}du = \frac{u^{-1/3+1}}{-1/3+1} + C$

$= \frac{3}{2}u^{2/3} + C = \boxed{\frac{3}{2}(\ln x)^{2/3} + C}$

(g) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ $\left| \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}dx \end{array} \Rightarrow \frac{dx}{\sqrt{x}} = 2du \right| = \displaystyle\int 2\cos u\, du$

$= 2\sin u + C = \boxed{2\sin\sqrt{x} + C}$

(h) $\displaystyle\int_0^1 (4x^3+1)(x^4+x)^5 dx$ $\left| \begin{array}{l} u = x^4+x \\ du = (4x^3+1)dx \\ 0 \to 0^4+0 = 0 \\ 1 \to 1^4+1 = 2 \end{array} \right| = \displaystyle\int_0^2 u^5 du = \frac{u^6}{6}\Big|_0^2 = \frac{2^6}{6} = \frac{64}{6} = \boxed{\frac{32}{3}}$

(i) $\cancel{\int \frac{dx}{x\sqrt[3]{\ln x}}}$

(j) $\int \overbrace{x^5}^{x^2 \cdot x^3} \sqrt{4+x^3}dx$

$$\left| \begin{array}{l} u = 4+x^3 \Rightarrow x^3 = u-4 \\ du = 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3} \end{array} \right| = \int x^2 \cdot x^3 \sqrt{4+x^3}\, dx$$

$$= \int (u-4)\sqrt{u}\, \frac{du}{3} = \frac{1}{3}\int (u \cdot u^{1/2} - 4u^{1/2})\, du$$

$$= \frac{1}{3}\int (u^{3/2} - 4u^{1/2})\, du = \frac{1}{3}\left(\frac{u^{3/2+1}}{3/2+1} - 4\frac{u^{1/2+1}}{1/2+1}\right) + C$$

$$= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - 4 \cdot \frac{2}{3}u^{3/2}\right) + C$$

$$= \boxed{\frac{1}{3}\left(\frac{2}{5}(4+x^3)^{5/2} - \frac{8}{3}(4+x^3)^{3/2}\right) + C}$$

(k) $\int_0^4 \frac{x}{\sqrt{1+2x}}dx$

$$\left| \begin{array}{l} u = 1+2x \Rightarrow x = \frac{u-1}{2} \\ du = 2dx \Rightarrow dx = \frac{du}{2} \\ x=0 \Rightarrow u = 1+2(0) = 1 \\ x=4 \Rightarrow u = 1+2(4) = 9 \end{array} \right| = \int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}} \frac{du}{2} = \frac{1}{4}\int_1^9 \frac{(u-1)}{\sqrt{u}}\, du$$

$$= \frac{1}{4}\int_1^9 (u-1)u^{-1/2}\, du = \frac{1}{4}\int_1^9 (u \cdot u^{-1/2} - u^{-1/2})\, du$$

$$= \frac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\, du$$

$$= \frac{1}{4}\left[\frac{u^{1/2+1}}{1/2+1} - \frac{u^{-1/2+1}}{-1/2+1}\right]_1^9 = \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)_1^9$$

$$= \frac{1}{4}\left(\frac{2}{3}9^{3/2} - 2 \cdot 9^{1/2} - \frac{2}{3} + 2\right) = \frac{1}{4}\left(\frac{2}{3} \cdot 27 - 2(3) + \frac{4}{3}\right)$$

$$= \frac{1}{4}\left(9 \cdot 2 - 6 + \frac{4}{3}\right) = \boxed{\frac{1}{4}\left(12 + \frac{4}{3}\right)} = \boxed{\frac{10}{3}}$$

4. If f is continuous and $\int_0^8 f(u)du = 3$, find $\int_0^2 x^2 f(x^3)dx$ by making an appropriate substitution.

$$\int_0^2 x^2 f(x^3)\, dx \quad \left| \begin{array}{l} u = x^3 \\ du = 3x^2 dx \\ x=0 \to u = 0^3 = 0 \\ x=2 \to u = 2^3 = 8 \end{array} \right| = \int_0^8 f(u)\frac{du}{3} = \frac{1}{3}\underbrace{\int_0^8 f(u)du}_{3}$$

$$= \frac{1}{3}(3) = \boxed{1}$$