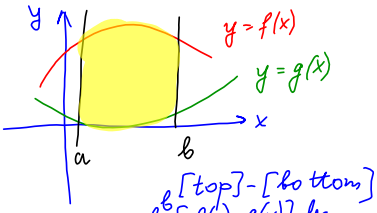
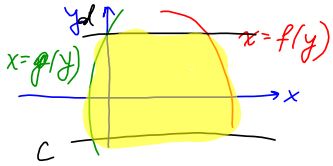


$$A = \int_a^b [\text{top}] - [\text{bottom}] \; [f(x) - g(x)]\,dx$$

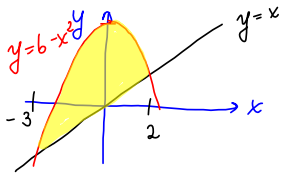


$$A = \int_c^d [\text{right}] - [\text{left}] \; [f(y) - g(y)]\,dy$$

Math 152/172 **WEEK in REVIEW 2** **Spring 2017**

Sections 7.1, 7.2, 7.3

1. Find the area of the region bounded by the line $y = x$ and the parabola $y = 6 - x^2$.



Points of intersection:

$6 - x^2 = x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

$-3 \le x \le 2$

$$A = \int_{-3}^{2} (6 - x^2 - x)\,dx$$

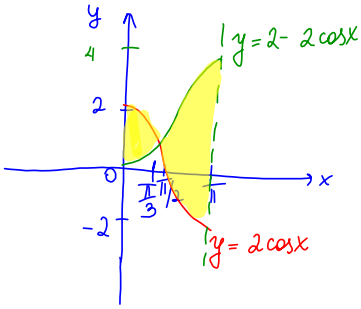
$$= \left(6x - \frac{x^3}{3} - \frac{x^2}{2}\right)_{-3}^{2}$$

$$= 6(2) - \frac{8}{3} - \frac{4}{2} - 6(-3) + \frac{(-3)^3}{3} + \frac{(-3)^2}{2}$$

$$= 12 - \frac{8}{3} - 2 + 18 - 9 + \frac{9}{2} = 19 + \frac{-16+27}{6}$$

$$= 19 + \frac{11}{6} = \boxed{\frac{2177}{6}}$$

2. Find the area of the region between $x = 2\cos x$ and $x = 2 - 2\cos x$, $0 \le x \le \pi$



Point of intersection:

$2\cos x = 2 - 2\cos x$

$\frac{4\cos x}{2} = \frac{2}{2}$

$2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

$$A = \int_0^{\pi/3} \left(2\cos x - (2 - 2\cos x)\right) dx + \int_{\pi/3}^{\pi} \left(2 - 2\cos x - 2\cos x\right) dx$$

$$= \int_0^{\pi/3} (4\cos x - 2)\, dx + \int_{\pi/3}^{\pi} (2 - 4\cos x)\, dx$$

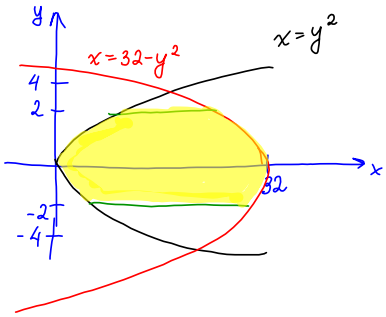
$$= \left[4\sin x - 2x\right]_0^{\pi/3} + \left(2x - 4\sin x\right)_{\pi/3}^{\pi}$$

$$= 4\sin\frac{\pi}{3} - \frac{2\pi}{3} + 2\pi - \frac{2\pi}{3} + 4\sin\frac{\pi}{3}$$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= \boxed{8\frac{\sqrt{3}}{2} + \frac{2\pi}{3}}$$

3. Find the area of the region between $x = y^2$ and $x = 32 - y^2$ from $y = -2$ to $y = 2$.



Points of intersection:

$y^2 = 32 - y^2$

$2y^2 = 32$

$y^2 = 16$

$y = \pm 4$

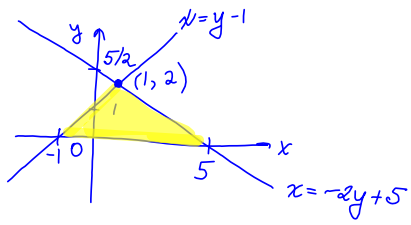
$$A = \int_{-2}^{2} (32 - y^2 - y^2)\, dy$$

$$= 2\int_0^2 (32 - 2y^2)\, dy$$

$$= 2\left(32y - \frac{2y^3}{3}\right)\Big|_0^2$$

$$= 2\left(32(2) - \frac{2(8)}{3}\right) = 2\left(64 - \frac{16}{3}\right) = \boxed{\frac{352}{3}}$$

4. Find the area of the region between lines $\overset{y=\frac{5-x}{2}}{x = -2y + 5}$, $\overset{y=x+1}{x = y - 1}$ and $\underbrace{y = 0}_{x\text{-axis}}$.



Point of intersection:

$-2y+5 = y-1$

$6 = 3y \Rightarrow y = 2 \Rightarrow x = y-1 = 1$

$$A = \int_{-1}^{1} (x+1)\,dx + \int_{1}^{5} \left(\frac{5}{2} - \frac{x}{2}\right) dx$$

$$= \left(\frac{x^2}{2} + x\right)_{-1}^{1} + \left(\frac{5}{2}x - \frac{x^2}{4}\right)_{1}^{5}$$

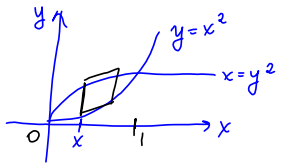
$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 + \frac{5}{2}\cdot 5 - \frac{25}{4} - \frac{5}{2} + \frac{1}{4} = \ldots$$

$$A = \int_0^2 [(5-2y)-(y-1)]\,dy = \int_0^2 (6-3y)\,dy$$

$$= \left[6y - \frac{3y^2}{2}\right]_0^2 = 12 - \frac{3(4)}{2} = \boxed{6}$$

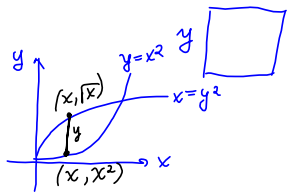
$V = \int_a^b A(x)\,dx$, where $A(x)$ is the area of a moving cross-section

5. The base of a certain solid is the region in the xy−plane bounded by the parabolas $y = x^2$ and $x = y^2$. Find the volume of this solid if every cross section perpendicular to the x−axis is a square with base in the xy−plane.



Integrate for x.

$0 \le x \le 1$



$A = y^2$

$y = [\text{top}] - [\text{bottom}]$

$y = \sqrt{x} - x^2$

$A(x) = y^2 = (\sqrt{x} - x^2)^2$

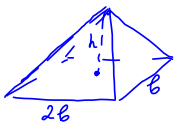
$= x - 2\sqrt{x}\,x^2 + x^4$

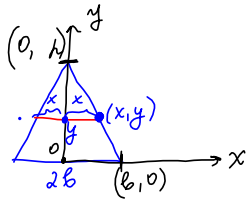
$A(x) = x - 2x^{5/2} + x^4$

$$V = \int_0^1 A(x)\,dx = \int_0^1 (x - 2x^{5/2} + x^4)\,dx$$

$$= \left[\frac{x^2}{2} - 2 \cdot \frac{x^{5/2+1}}{5/2+1} + \frac{x^5}{5}\right]_0^1 = \boxed{\frac{1}{2} - \frac{4}{7} + \frac{1}{5}}$$

6. Find the volume of a pyramid with height h and rectangular base with dimensions b and $2b$.





Equation of the line through $(b,0)$ and $(0,h)$

slope $= -\frac{h}{b}$

$$y - h = -\frac{h}{b}\cdot x$$

$$x = -\frac{b}{h}(y-h)$$

$$x = -\frac{b}{h}y + b$$

Integrate for y, $0 \le y \le h$.

Pick $0 \le y \le h$, do the plain perpendicular to the y-axis through y

Cross-section is a rectangle with dimensions $2x$ and x

$$A = (2x)(x) = 2x^2$$

$$A(y) = 2\left(b - \frac{b}{h}y\right)^2 = 2b^2\left(1 - \frac{1}{h}y\right)^2$$

$$V = \int_0^h A(y)\,dy = 2b^2\int_0^h \left(1 - \frac{1}{h}y\right)^2 dy$$

$$= 2b^2\int_0^h \left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right) dy$$

$$= 2b^2\left[y - \frac{\cancel{2}y^2}{\cancel{2}h} + \frac{1}{h^2}\frac{y^3}{3}\right]_0^h$$

$$= 2b^2\left(h - \frac{h^{\cancel{2}}}{\cancel{h}} + \frac{h^{\cancel{3}}}{3\cancel{h^2}}\right)$$

$$= \boxed{\frac{2}{3}b^2h}$$

washers.

$$V_x = \pi \int_a^b \left([\text{outer radius}]^2 - [\text{inner radius}]^2 \right) dx$$

$$V_y = \pi \int_c^d \left([\text{outer radius}]^2 - [\text{inner radius}]^2 \right) dy$$

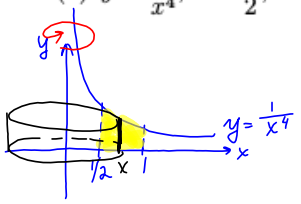
cylindrical shells:

$$V_x = 2\pi \int_a^b [\text{radius}][\text{height}]\, dy$$

$$V_y = 2\pi \int_c^d [\text{radius}][\text{height}]\, dx$$

7. Find the volume generated by rotating the region bounded by the given curves about the specified line.

(a) $y = \dfrac{1}{x^4}$, $x = \dfrac{1}{2}$, $x = 1$, $\underbrace{y = 0}_{x\text{-axis}}$, about the y-axis



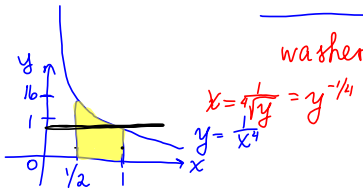
shells: integrate for x, $\frac{1}{2} \le x \le 1$

$[\text{height}] = \frac{1}{x^4}$

$[\text{radius}] = x$

$$V_y = 2\pi \int_{1/2}^{1} x \cdot \frac{1}{x^4}\, dx = 2\pi \int_{1/2}^{1} \frac{1}{x^3}\, dx = 2\pi \left. \frac{x^{-3+1}}{-3+1} \right]_{1/2}^{1}$$

$$= 2\pi \cdot \left. \frac{x^{-2}}{-2} \right]_{1/2}^{1} = \pi(-1+4) = \boxed{3\pi}$$



washers: integrate for y

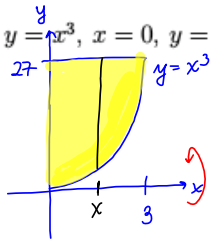
$0 \le y \le 1$	$1 \le y \le 16$
$[\text{outer radius}] = 1$	$[\text{outer radius}] = y^{-1/4}$
$[\text{inner radius}] = 1/2$	$[\text{inner radius}] = 1/2$

$$V_y = \pi \int_0^1 \left(1 - \frac{1}{4}\right) dy + \pi \int_1^{16} \left([y^{-1/4}]^2 - \frac{1}{4} \right) dy$$

$$= \pi \cdot \frac{3}{4} y \Big]_0^1 + \pi \int_1^{16} \left(y^{-1/2} - \frac{1}{4} \right) dy = \frac{3\pi}{4} + \pi \left(\frac{y^{1/2}}{1/2} - \frac{1}{4} y \right)_1^{16}$$

$$= \frac{3\pi}{4} + \pi \left(2\sqrt{y} - \frac{1}{4} y \right) = \frac{3\pi}{4} + \pi \left(8 - 4 - 2 + \frac{1}{4} \right) = \boxed{3\pi}$$

(b) $y = x^3$, $x = 0$, $y = 27$, about the x-axis



washers:

$0 \le x \le 3$

[outer radius] $= 27$

[inner radius] $= x^3$

$$V_x = \pi \int_0^3 \left(27^2 - (x^3)^2 \right) dx$$

$$= \pi \int_0^3 \left(729 - x^6 \right) dx = \pi \left(729x - \frac{x^7}{7} \right)\Big|_0^3$$

$$= \pi \left(729(3) - \frac{3^7}{7} \right)$$

$$= \boxed{\pi \left(2,187 - \frac{2,187}{7} \right)}$$

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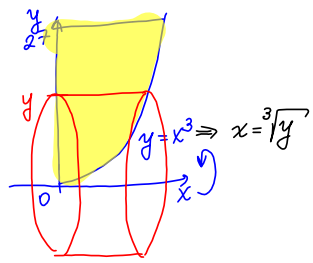
shells: integrate for y, $0 \le y \le 27$



[radius] $= y$

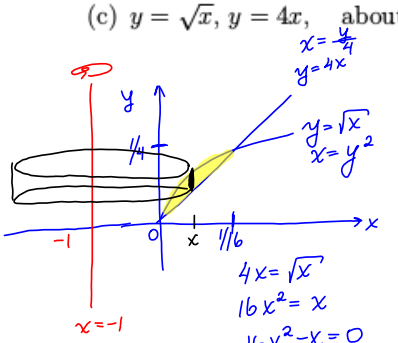
[height] $= \sqrt[3]{y}$

$$V_x = 2\pi \int_0^{27} y \sqrt[3]{y}\, dy$$

$$= 2\pi \int_0^{27} y^{4/3}\, dy = 2\pi \left. \frac{y^{4/3+1}}{4/3+1} \right]_0^{27}$$

$$= 2\pi \cdot \frac{3}{7} y^{7/3} \Big]_0^{27} = \frac{6\pi}{7} 3^7 = \boxed{\frac{6\pi}{7}(2,187)}$$

(c) $y = \sqrt{x}$, $y = 4x$, about $\overbrace{x = -1}^{\text{parallel to the } y\text{-axis.}}$



$4x = \sqrt{x}$

$16x^2 = x$

$16x^2 - x = 0$

$x(16x - 1) = 0$

$x_1 = 0$, $x_2 = \frac{1}{16}$

$y = 0$; $y = 4x_2 = \frac{1}{4}$

shells.

integrate for x

$0 \le x \le \frac{1}{16}$

[radius] $= x + 1$

[height] $= \sqrt{x} - 4x$

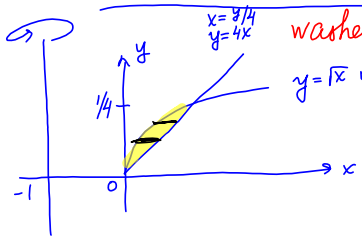
$$V = 2\pi \int_0^{1/16} (x+1)(\sqrt{x} - 4x)\, dx$$

$$= 2\pi \int_0^{1/16} \left(\cancel{x\sqrt{x}}^{\,x^{3/2}} - 4x^2 + \cancel{\sqrt{x}}^{\,x^{1/2}} - 4x \right) dx$$

$$= 2\pi \left(\frac{x^{5/2}}{5/2} - \frac{4x^3}{3} + \frac{x^{3/2}}{3/2} - \frac{4x^2}{2} \right)_0^{1/16}$$

$$= 2\pi \left(\frac{2}{5}\left(\frac{1}{16}\right)^{5/2} - \frac{4}{3}\left(\frac{1}{16}\right)^3 + \frac{2}{3}\left(\frac{1}{16}\right)^{3/2} - 2\left(\frac{1}{16}\right)^2 \right)$$

$$= \boxed{2\pi \left(\frac{2}{5}(1024) - \frac{4}{3}\left(\frac{1}{4096}\right) + \frac{2}{3}\,\frac{1}{64} - \frac{2}{256} \right)}$$



washers

$y = \sqrt{x}$ or $x = y^2$

integrate for y, $0 \le y \le \frac{1}{4}$.

[inner radius] $= 1 + y^2$

[outer radius] $= 1 + \frac{y}{4}$

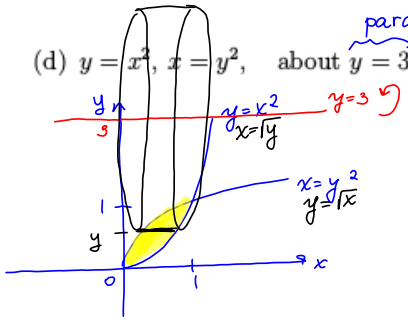
$$V = \pi \int_0^{1/4} \left[\left(1 + \frac{y}{4}\right)^2 - (1 + y^2)^2 \right] dy$$

$$= \pi \int_0^{1/4} \left(\cancel{1} + \frac{y}{2} + \frac{y^2}{16} - \cancel{1} - 2y^2 - y^4 \right) dy$$

$$= \pi \left(\frac{y^2}{4} + \frac{y^3}{48} - \frac{2y^3}{3} - \frac{y^5}{5} \right)_0^{1/4}$$

$$= \boxed{\pi \left(\frac{1}{16 \cdot 4} + \frac{1}{64 \cdot 48} - \frac{2}{3} \cdot \frac{1}{64} - \frac{1}{5} \cdot \frac{1}{1024} \right)}$$

(d) $y = x^2$, $x = y^2$, about $y = 3$ parallel to the x-axis.



washers. integrate for x.
$0 \le x \le 1$

[inner radius] $= 3 - \sqrt{x}$
[outer radius] $= 3 - x^2$

$$V = \pi \int_0^1 \left[(3-x^2)^2 - (3-\sqrt{x})^2 \right] dx$$

$$= \pi \int_0^1 \left(\cancel{9} - 6x^2 + x^4 - \cancel{9} + 6\sqrt{x} - x \right) dx$$

$$= \pi \left(-\frac{6x^3}{3} + \frac{x^5}{5} + 6 \cdot x^{3/2} \cdot \frac{2}{3} - \frac{x^2}{2} \right)_0^1$$

$$= \pi \left(-2 + \frac{1}{5} + 4 - \frac{1}{2} \right) = \pi \left(2 - \frac{1}{2} + \frac{1}{5} \right)$$

$$= \pi \frac{20-5+2}{10} = \boxed{\frac{17\pi}{10}}$$

shells: integrate for y, $0 \le y \le 1$

[height] $= \sqrt{y} - y^2$
[radius] $= 3 - y$

$$V = 2\pi \int_0^1 (3-y)(\sqrt{y} - y^2)\, dy = 2\pi \int_0^1 \left(3\sqrt{y} - y^{3/2} - 3y^2 + y^3 \right) dy$$

$$= 2\pi \left(\cancel{3}\, y^{3/2} \cdot \frac{2}{\cancel{3}} - y^{5/2} \cdot \frac{2}{5} - \frac{\cancel{3}y^3}{\cancel{3}} + \frac{y^4}{4} \right)_0^1$$

$$= 2\pi \left(2 - \frac{2}{5} - 1 + \frac{1}{4} \right) = 2\pi \left(\frac{20-8+5}{20} \right) = \boxed{\frac{17\pi}{10}}$$