

$$W = \int_a^b f(x) dx$$

Math 152/172

**WEEK in REVIEW 3**  
Sections 7.4, 7.5, 8.1

Spring 2017

1. A force of  $F(x) = x^4 - \sin(4\pi x) + 12$ , where  $x$  is in meters, acts on an object. What is the work required to move the object from  $x = 3$  to  $x = 5$ ?

$$\begin{aligned}
 W &= \int_3^5 (x^4 - \sin(4\pi x) + 12) dx = \left( \frac{x^5}{5} + 12x \right) \Big|_3^5 - \int_{12\pi}^{20\pi} \sin u du \\
 &\quad \left| \begin{array}{l} u = 4\pi x \\ du = 4\pi dx \Rightarrow dx = \frac{du}{4\pi} \\ x = 3 \Rightarrow u = 4\pi(3) = 12\pi \\ x = 5 \Rightarrow u = 4\pi(5) = 20\pi \end{array} \right| \\
 &= \frac{5^5}{5} + 12(5) - \frac{3^5}{5} - 12(3) + \cos u \Big|_{12\pi}^{20\pi} \\
 &= 625 + 60 - \frac{243}{5} - 36 + \cos(20\pi) - \cos(12\pi) \\
 &= \boxed{605 - \frac{243}{5}} \quad (\text{J})
 \end{aligned}$$

$$F(x) = kx$$

*k is a constant*

2. If the force required to stretch a spring 3 ft beyond its natural length is 12 lb, how much work is needed to stretch it 46 inches beyond its natural length?

$$F(3) = 12 \Rightarrow k = 4$$

$46 \text{ inches} = \frac{46}{12} \text{ ft}$

$$W = \int_0^{\frac{46}{12}} 4x \, dx = 2x^2 \Big|_0^{\frac{23}{6}} = 2 \left(\frac{23}{6}\right)^2 = \frac{529}{18} (\text{ft-lbs})$$

3. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

*unstretched*  $\downarrow$   $\begin{array}{c} 3 \\ | \\ 3 \\ | \\ 20 \text{ cm} \rightarrow 0 \\ | \\ 25 \text{ cm} = 25 - 20 = 5 \text{ cm} \\ = 0.05 \text{ m} \end{array}$

$$10 = \int_0^{0.05} kx \, dx = \frac{kx^2}{2} \Big|_0^{0.05}$$

$$10 = \frac{k}{2} \cdot 0.0025$$

$$20 = k \cdot \frac{25}{10000}$$

$$20 = \frac{k}{400} \Rightarrow k = 8000$$

$$f(x) = 8000x$$

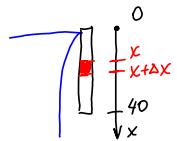
$$30 \text{ cm} \rightarrow 30 - 20 = 10 \text{ cm} = 0.1 \text{ m}$$

$$80 \text{ cm} \rightarrow 80 - 20 = 60 \text{ cm} = 0.6 \text{ m}$$

$$W = \int_{0.1}^{0.6} 8000x \, dx = 8000 \frac{x^2}{2} \Big|_{0.1}^{0.6} = 4000 (0.6^2 - 0.1^2)$$

$$= 4000 \cdot \left( \frac{36}{100} - \frac{1}{100} \right) = (40)(35) = \boxed{1400 \text{ (J)}}$$

4. A heavy rope 40 ft long, weighs 0.4 lb/ft and hangs over the edge of a tall building. How much work is done in pulling the rope to the top of the building?



$$0 \leq x \leq 40$$

Take a part of the rope between  $x$  and  $x + \Delta x$

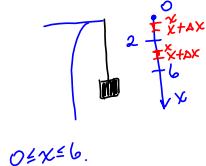
$$[\text{weight}] = 0.4 \Delta x$$

$$[\text{distance traveled}] = x$$

$$W = \int_0^{40} [\text{weight}][\text{distance}] dx$$

$$= \int_0^{40} 0.4x dx = 0.2 x^2 \Big|_0^{40} = 0.2(1600) = \boxed{320 \text{ ft-lbs}}$$

5. A uniform cable hanging over the edge of a tall building is 6 m long and weighs 20 kg. If 25 kg weight is attached to the cable, how much work is required to pull 2 m of the cable to the top of the building?



$$0 \leq x \leq 6$$

$$\text{Weight of the rope} = \frac{6}{20} \cdot g, \text{ where } g = 9.8 \text{ m/sec}^2$$

$$\text{Total work } W = \underbrace{w_1}_{\substack{\text{weight} \\ \text{first}}} + \underbrace{w_2}_{\substack{\text{rest of} \\ \text{the rope}}} + \underbrace{w_3}_{\substack{\text{2m of} \\ \text{the rope}}}$$

$$w_1 = (25g)2 = 50g (\text{J})$$

$\underbrace{0 \leq x \leq 2}_{\text{Take a part of the rope between } x \text{ and } x + \Delta x}$

$$[\text{weight}] = \frac{3}{10} g \Delta x$$

$$[\text{distance}] = x$$

$$w_2 = \int_0^2 \frac{3}{10} g x dx = \frac{3g}{10} \frac{x^2}{2} \Big|_0^2 = \frac{3g}{5} (\text{J})$$

$\underbrace{2 \leq x \leq 6}_{\text{Take a piece of the rope between } x \text{ and } x + \Delta x}$

$$[\text{weight}] = \frac{3}{10} g \Delta x$$

$$[\text{distance}] = 2$$

$$w_3 = \int_2^6 \frac{3}{10} g (2) dx = \frac{3}{10} g 2x \Big|_2^6 = \frac{3}{5} g (6-2) = \frac{12g}{5} (\text{J})$$

$$\text{total work } w = 50g + \frac{3g}{5} + \frac{12g}{5} = \boxed{53g (\text{J})}$$

$$w = \int_{6-2}^6 \frac{3}{10} g x dx$$

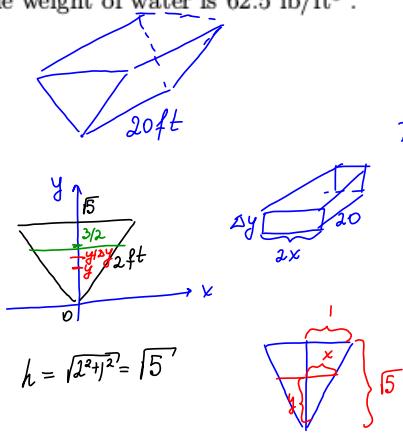
$$= \frac{3g}{10} \frac{x^2}{2} \Big|_{6-2}^6$$

$$= \frac{3g}{20} (36 - 16) = \boxed{3g (\text{J})}$$

shortcut: work done by pulling up a ft of a rope of length L weighing  $g \text{ lb/ft}$  (assume  $a < L$ ) is

$$W = \int_a^L g x dx$$

6. A tank of water is 20 ft long and has a vertical cross section in a shape of an equilateral triangle with sides 2 ft long. The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. The weight of water is 62.5 lb/ft<sup>3</sup>.



$0 \leq y \leq \frac{18}{12} = \frac{3}{2}$   
 Pick  $y$  between 0 and  $\frac{3}{2}$ .  
 Take a "slice" of water between  $y$  and  $y + \Delta y$   
 Volume of the "slice" =  $2x(20)\Delta y$   
 express  $x$  in terms of  $y$ .  
 similar triangles:

$$\frac{1}{x} = \frac{\sqrt{15}}{y}$$

$$x = \frac{y}{\sqrt{15}}$$

$$[\text{volume}] = \frac{2y}{\sqrt{15}} (20) \Delta y = \frac{40y}{\sqrt{15}} \Delta y$$

$$[\text{weight}] = [\text{volume}] (62.5) = \frac{40y}{\sqrt{15}} (62.5)$$

$$[\text{distance}] = \sqrt{15} - y$$

$$W = \int_0^{3/2} \frac{40y}{\sqrt{15}} (62.5)(\sqrt{15} - y) dy = \frac{62.5(40)}{\sqrt{15}} \int_0^{3/2} y(\sqrt{15} - y) dy$$

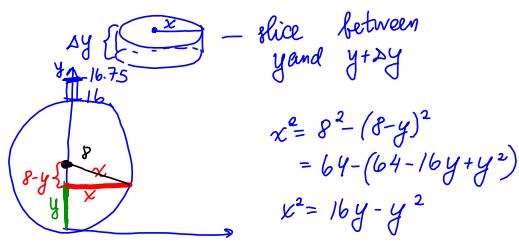
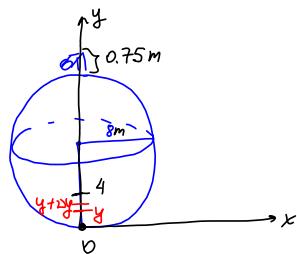
$$= \frac{62.5(40)}{\sqrt{15}} \int_0^{3/2} (\sqrt{15}y - y^2) dy$$

$$= \frac{62.5(40)}{\sqrt{15}} \left( \frac{\sqrt{15}y^2}{2} - \frac{y^3}{3} \right)_0^{3/2}$$

$$= \boxed{\frac{62.5(40)}{\sqrt{15}} \left( \frac{\sqrt{15}}{2} \cdot \frac{9}{4} - \frac{1}{3} \cdot \frac{27}{8} \right) (\text{ft-lbs})}$$

7. A spherical tank with radius 8 m is half full of water. The water pumped out of the spout of the top of the tank that is 75 cm high. Find the work done needed to pump out water through the spout until the water level is 4 m from the bottom. The density of water is  $1000 \text{ kg/m}^3$ .

$$[\text{weight of water}] = g \cdot 10^3, \text{ where } g = 9.8 \\ = 9800 \text{ N}$$



$$16 \times 16 = 256 \\ 16 \cdot 0.75 = 16 \cdot \frac{3}{4} = 12$$

$$0 \leq y \leq 4.$$

express  $x$  in terms of  $y$ .

$$[\text{volume}] = \pi x^2 \Delta y \\ = \pi (16y - y^2) \Delta y$$

$$[\text{weight}] = \pi (16y - y^2) \Delta y (9800)$$

$$[\text{distance}] = 16 - y + 0.75 \\ = 16.75 - y$$

$$W = \int_0^4 \pi (16y - y^2) (9800) (16.75 - y) dy$$

$$= 9800 \pi \int_0^4 (16y - y^2)(16.75 - y) dy$$

$$= 9800 \pi \int_0^4 (16(16.75)y - 16y^2 - 16.75y^2 + y^3) dy$$

$$= 9800 \pi \int_0^4 (268y - 32.75y^2 + y^3) dy$$

$$= 9800 \pi \left[ \frac{268y^2}{2} - 32.75 \frac{y^3}{3} + \frac{y^4}{4} \right]_0^4$$

$$= \boxed{9800 \pi (536 - 32.75 \cdot \frac{4}{3} + 64) (J)}$$

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

8. Determine the average value of the function  $f(x) = \frac{\sqrt{3}-1}{1+x^2}$  over the interval  $[1, \sqrt{3}]$ .

$$\begin{aligned} f_{ave} &= \frac{1}{\sqrt{3}-1} \int_1^{\sqrt{3}} \frac{\sqrt{3}-1}{1+x^2} dx = \int_1^{\sqrt{3}} \frac{dx}{1+x^2} \\ &= \arctan x \Big|_1^{\sqrt{3}} = \arctan \sqrt{3} - \arctan 1 \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}} \end{aligned}$$

9. Find the value(s)  $a$  such that the average value of the function  $f(x) = 3x^2 - 2x - 3$  over the interval  $[a, 0]$  is equal to 2. ( $a < 0$ )

$$\begin{aligned} f_{ave} &= \frac{1}{a+0} \int_a^0 (3x^2 - 2x - 3) dx \\ &= \frac{1}{a} \left( x^3 - x^2 - 3x \right) \Big|_a^0 = -\frac{1}{a} (-a^3 + a^2 + 3a) \\ &= a^2 - a - 3 = 2 \\ &= a^2 - a - 5 = 0 \\ a_1 &= \frac{1 + \sqrt{1 - 4(-5)}}{2} = \frac{1 + \sqrt{21}}{2} > 0 \text{ - not valid} \\ a_2 &= \boxed{\frac{1 - \sqrt{21}}{2}} \end{aligned}$$

Integration by parts:  $\int u v' dx = uv - \int vu' dx$

$$\int_a^b u v' dx = uv|_a^b - \int_a^b vu' dx$$

10. Find the integral.

$$(a) \int_0^\pi x^2 \sin x dx = (x^2 \cos x - 2x(-\sin x) + 2 \cos x)|_0^\pi = (-x^2 \cos x + 2x \sin x + 2 \cos x)|_0^\pi$$

$$\int x^n \left\{ \begin{array}{l} e^{ax} \\ \sin bx \\ \cos cx \end{array} \right\} dx = \left| \begin{array}{l} u = x^n \\ v' = \left\{ \begin{array}{l} e^{ax} \\ \sin bx \\ \cos cx \end{array} \right\} \end{array} \right|$$

Tabulation method:

D	I
$x^2$	$\sin x$
$-2x$	$-\cos x$
$2$	$-\sin x$
$0$	$\cos x$

$$\begin{aligned} &= -\pi^2 \cos \pi + 2\pi \sin \pi + 2 \cos \pi \\ &\quad - 2 \cos 0 \\ &= \boxed{\pi^2 - 4} \end{aligned}$$

$$(b) \int (\ln x)^2 dx = \left| \begin{array}{l} u = (\ln x)^2 \\ u' = 2 \ln x \frac{1}{x} \end{array} \right. \left| \begin{array}{l} v' = 1 \\ v = x \end{array} \right. = x(\ln x)^2 - \int 2 \ln x \frac{1}{x} x dx$$

$$\int x^n \left\{ \begin{array}{l} \ln x \\ \arctan x \\ \arcsin x \end{array} \right\} dx = \left| \begin{array}{l} u = \ln x \\ u' = \frac{1}{x} \\ \arctan x \\ \arcsin x \end{array} \right. \left| \begin{array}{l} v' = x^n \\ v = x \end{array} \right. = x(\ln x)^2 - 2 \int \ln x dx \left| \begin{array}{l} u = \ln x \\ u' = \frac{1}{x} \end{array} \right. = x(\ln x)^2 - 2(x \ln x - \int \frac{1}{x} x dx) = x(\ln x)^2 - 2(x \ln x - \int dx) = \boxed{x(\ln x)^2 - 2x \ln x + 2x + C}$$

$$(c) \int e^{-t} \cos(3t) dt \quad \text{NEXT slide}$$

$$\begin{aligned}
 (c) \quad & \int e^{-t} \cos(3t) dt \quad \left| \begin{array}{l} u = \cos(3t) \\ u' = -3\sin(3t) \end{array} \right. \quad \left| \begin{array}{l} v' = e^{-t} \\ v = -e^{-t} \end{array} \right. \\
 &= -e^{-t} \cos(3t) - \int (+3\sin(3t)) (+e^{-t}) dt \\
 &= -e^{-t} \cos(3t) - 3 \int \sin(3t) e^{-t} dt \quad \left| \begin{array}{l} u = \sin(3t) \\ u' = 3\cos(3t) \end{array} \right. \quad \left| \begin{array}{l} v' = e^{-t} \\ v = -e^{-t} \end{array} \right. \\
 &= -e^{-t} \cos(3t) + 3 \left[ +e^{-t} \sin(3t) + \int 3\cos(3t)(-e^{-t}) dt \right] \\
 \underbrace{\int e^{-t} \cos(3t) dt}_I &= -e^{-t} \cos(3t) + 3e^{-t} \sin(3t) - 9 \underbrace{\int e^{-t} \cos(3t) dt}_I \\
 I &= -e^{-t} \cos(3t) + 3e^{-t} \sin(3t) - 9I \\
 10I &= e^{-t}(3\sin 3t - \cos 3t) \\
 I &= \left[ \frac{e^{-t}}{10} (3\sin 3t - \cos 3t) + C \right]
 \end{aligned}$$