

Section 8.2 Trigonometric Integrals.

$$\bullet \int \sin^m x \cos^n x \, dx$$

1. if the power of cosine is odd, do the substitution $u = \sin x$ (save one factor of $\cos x$ and convert the rest to sine)
2. if the power of sine is odd, do the substitution $u = \cos x$ (save one factor of $\sin x$ and convert the rest to cosine)
3. if both m and n are even, use half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \sin x \cos x = \frac{1}{2} \sin 2x$$

$$\bullet \int \tan^m x \sec^n x \, dx$$

1. if the power of secant is even, do the substitution $u = \tan x$ (save one factor of $\sec^2 x$ and convert the rest to tangent)
2. if the power of tangent is odd, do the substitution $u = \sec x$ (save one factor of $\sec x \tan x$ and convert the rest to secant)

$$\bullet \int \sin mx \cos nx \, dx, \int \sin mx \sin nx \, dx, \int \cos mx \cos nx \, dx$$

use the corresponding identity:

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

Examples. Find the integral

$$1. \int \sin^3 x \cos^4 x \, dx = \int \sin x \overbrace{\sin^2 x}^{1-\cos^2 x} \cos^4 x \, dx = \int \sin x (1-\cos^2 x) \cos^4 x \, dx \quad \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right|$$

$$= - \int (1-u^2) u^4 \, du = - \int (u^4 - u^6) \, du = - \frac{u^5}{5} + \frac{u^7}{7} + C \quad \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$$

$$2. \int_0^{\pi/8} \sin^2(2x) \cos^5(2x) \, dx = \int_0^{\pi/8} \sin^2(2x) \overbrace{\cos(2x)}^{[1-\sin^2(2x)]^{1/2}} \overbrace{\cos^4(2x)}^{[\cos^2(2x)]^2} \, dx = \int_0^{\pi/8} \sin^2(2x) \cos(2x) [1 - \sin^2(2x)]^{1/2} \cos^2(2x) \, dx$$

$$\left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos(2x) \, dx \\ \cos(2x) \, dx = \frac{du}{2} \\ x=0 \Rightarrow u=\sin 0=0 \\ x=\frac{\pi}{8} \Rightarrow u=\sin \frac{\pi}{4}=\frac{\sqrt{2}}{2} \\ = \frac{\sqrt{2}}{2} \end{array} \right.$$

$$= \int_0^{\sqrt{2}/2} u^2 (1-u^2)^2 \frac{du}{2} = \frac{1}{2} \int_0^{\sqrt{2}/2} u^2 (1-2u^2+u^4) \, du = \frac{1}{2} \int_0^{\sqrt{2}/2} (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{2} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) \Big|_0^{\sqrt{2}/2} = \frac{1}{2} \left(\frac{1}{3} \left(\frac{\sqrt{2}}{2} \right)^3 - \frac{2}{5} \left(\frac{\sqrt{2}}{2} \right)^5 + \frac{1}{7} \left(\frac{\sqrt{2}}{2} \right)^7 \right) \boxed{\frac{1}{2} \left(\frac{1}{3} \cdot \frac{\sqrt{2}}{4} - \frac{2}{5} \cdot \frac{\sqrt{2}}{8} + \frac{1}{7} \cdot \frac{\sqrt{2}}{16} \right)}$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + C \quad \int \sin bx dx = -\frac{1}{b} \cos bx + C$$

$$\frac{1}{2} \sin 2x \quad \frac{1+\cos 2x}{2}$$

$$3. \int \sin^2 x \cos^4 x dx = \int (\sin^2 x \cos^2 x) \cos^2 x dx = \int (\sin x \cos x)^2 \cos^2 x dx = \int \frac{1}{4} \sin^2(2x) \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{8} \int \sin^2(2x)(1+\cos 2x) dx = \frac{1}{8} \int \sin^2 2x dx + \frac{1}{8} \int \sin^2 2x \cos 2x dx$$

$$\left| \begin{array}{l} u = \sin 2x \\ du = 2 \cos 2x dx \\ \cos 2x dx = \frac{du}{2} \end{array} \right|$$

$$= \frac{1}{8} \int \frac{1-\cos 4x}{2} dx + \frac{1}{8} \int u^2 \frac{du}{2}$$

$$= \frac{1}{16} \left(x - \frac{1}{4} \sin 4x \right) + \frac{1}{48} \sin^3 2x + C = \boxed{\frac{1}{16} \left(x - \frac{1}{4} \sin 4x \right) + \frac{1}{48} \sin^3 2x + C}$$

$$4. \int_0^{\pi/4} \tan^4 x \sec^4 x dx = \int_0^{\pi/4} \tan^4 x \underbrace{\sec^2 x}_{\tan^2 x + 1} \sec^2 x dx = \int_0^{\pi/4} \tan^4 x (\tan^2 x + 1) \sec^2 x dx$$

$$\left| \begin{array}{l} u = \tan x \\ du = \sec^2 x dx \\ x=0 \Rightarrow u=\tan 0=0 \\ x=\pi/4 \Rightarrow u=\tan \frac{\pi}{4}=1 \end{array} \right|$$

$$= \int_0^1 u^4 (u^2 + 1) du = \int_0^1 (u^6 + u^4) du = \left(\frac{u^7}{7} + \frac{u^5}{5} \right) \Big|_0^1 = \frac{1}{7} + \frac{1}{5} = \boxed{\frac{12}{35}}$$

$$5. \int \tan^3 x \sec^3 x \, dx = \int (\tan x \sec x) \sec^2 x \overbrace{\tan^2 x}^{\sec^2 x - 1} \, dx = \int (\tan x \sec x) \sec^2 x (\sec^2 x - 1) \, dx \quad \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right.$$

$$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

$$6. \int \sin 3x \cos x \, dx$$

$$\sin A \cos B = \frac{1}{2} (\sin(A-B) + \sin(A+B))$$

$$= \int \frac{1}{2} (\sin(3x-x) + \sin(3x+x)) \, dx = \frac{1}{2} \int (\sin 2x + \sin 4x) \, dx$$

$$= \boxed{\frac{1}{2} (-\frac{1}{2} \cos 2x - \frac{1}{4} \cos 4x) + C}$$

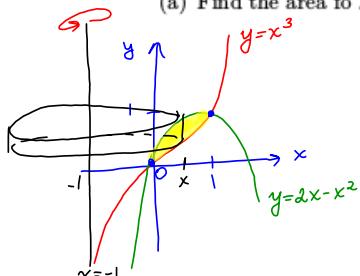
$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

Review for Test 1.

1. Let \mathcal{R} be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$.

(a) Find the area of \mathcal{R}



Points of intersection:

$$\begin{aligned} x^3 &= 2x - x^2 \\ x^3 + x^2 - 2x &= 0 \\ x(x^2 + x - 2) &= 0 \\ x(x+2)(x-1) &= 0 \\ x=0, x=-2, x=1 \end{aligned}$$

$$\begin{aligned} A &= \int_0^1 ([\text{top}] - [\text{bottom}]) dx = \int_0^1 (2x - x^2 - x^3) dx \\ &= \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \boxed{\frac{5}{12}} \end{aligned}$$

- (b) Find the volume obtained by rotating \mathcal{R} about the line $x = -1$.

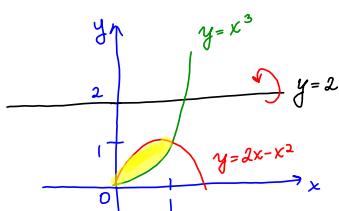
Shells.

$$[\text{radius}] = x+1$$

$$[\text{height}] = 2x - x^2 - x^3$$

$$V = 2\pi \int_0^1 (x+1)(2x - x^2 - x^3) dx = \dots$$

- (c) Find the volume obtained by rotating \mathcal{R} about the line $y = 2$.



Washers.

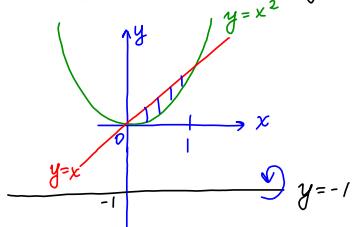
$$[\text{inner radius}] = 2 - (2x - x^2)$$

$$[\text{outer radius}] = 2 - x^3$$

$$\begin{aligned} V &= \pi \int_0^1 ([\text{outer radius}]^2 - [\text{inner radius}]^2) dx \\ &= \pi \int_0^1 [(2-x^3)^2 - (2-2x+x^2)^2] dx = \dots \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about

(a) the ~~the~~ line $y = -1$.



washers.

$$[\text{inner radius}] = x^2 + 1$$

$$[\text{outer radius}] = x + 1$$

$$V = \pi \int_0^1 ((x+1)^2 - (x^2+1)^2) dx = \dots$$

(b) the y -axis

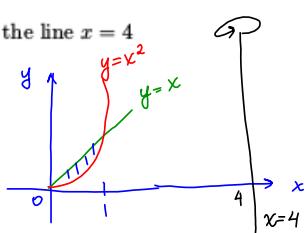
shells.

$$[\text{radius}] = x$$

$$[\text{height}] = [\text{top}] - [\text{bottom}] = x - x^2$$

$$V = 2\pi \int_0^1 x(x-x^2) dx = \dots$$

(c) the line $x = 4$

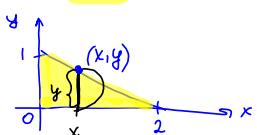


shells: $[\text{radius}] = 4 - x$

$$[\text{height}] = x - x^2$$

$$V = 2\pi \int_0^1 (4-x)(x-x^2) dx = \dots$$

3. The base of solid S is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$. Cross-sections perpendicular to the x -axis are semicircles. Find the volume of S .



similar triangles:

$$\frac{y}{1} = \frac{2-x}{2}$$

$$y = 1 - \frac{x}{2}$$

equation of the line through $(2,0)$ and $(0,1)$

$$\text{slope} = -\frac{1}{2}$$

$$y - 0 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{x}{2} + 1$$

Integrate for x , $0 \leq x \leq 2$.

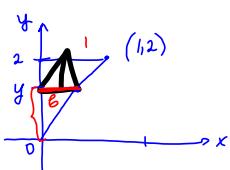
$$A = \frac{1}{2}\pi \left(\frac{y}{2}\right)^2 = \frac{\pi}{8}y^2$$

Express y in terms of x .

$$A(x) = \frac{\pi}{8} \left(1 - \frac{x}{2}\right)^2$$

$$V = \int_0^2 A(x) dx = \frac{\pi}{8} \int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \dots$$

6. Find the volume of the solid whose base is the triangular region with endpoints $(0,0)$, $(0,2)$, and $(1,2)$, and where cross sections perpendicular to the y -axis are isosceles triangles with height equal to base.



integrate for $0 \leq y \leq 2$.

$$A = \frac{1}{2} b h = \frac{1}{2} b^2$$

Express b in terms of y

$$\frac{b}{1} = \frac{y}{2} \Rightarrow b = \frac{y}{2}$$

$$A(y) = \frac{1}{2} \left(\frac{y}{2}\right)^2 = \frac{1}{8} y^2$$

$$V = \int_0^2 \frac{1}{8} y^2 dy = \dots$$



4. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high. How much work is done in pulling the rope to the top of the building?

$0 \leq x \leq 50$
 Take a piece of the rope between x and $x + \Delta x$
 [weight] = $0.5 \Delta x$
 [distance] = x
 $W = \int_0^{50} 0.5x \, dx$

Find the work done in pulling 10 ft of the rope.

$$W = \int_{50-10}^{50} 0.5x \, dx = \int_{40}^{50} 0.5x \, dx$$

Weight of 30 lb is attached to the rope. Find the work done in pulling the weight to the top of the building.



$$W = (30)(50) + \int_0^{50} 0.5x \, dx$$

5. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

20 cm is the natural length.

25 cm is 5 cm beyond the natural length.

$$5 \text{ cm} = 0.05 \text{ m}$$

$f(x) = kx$, k is an unknown const.

$$10 = \int_0^{0.05} kx \, dx$$

$$10 = \frac{kx^2}{2} \Big|_0^{0.05}$$

$$10 = \frac{0.0025k}{2} \Rightarrow k = \frac{200000}{0.0025} = \frac{100 \cdot 2000}{25} = 8000$$

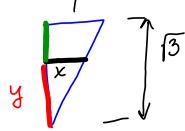
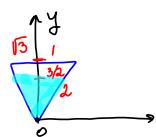
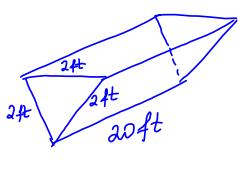
$$f(x) = 8000x$$

$$30 \rightarrow 30 - 20 = 10 \text{ cm} = 0.1 \text{ m}$$

$$80 \rightarrow 80 - 20 = 60 \text{ cm} = 0.6 \text{ m}$$

$$W = \int_{0.1}^{0.6} 8000x \, dx = 4000x^2 \Big|_{0.1}^{0.6} = 4000(0.36 - 0.01) \\ = 4000(0.35) = \boxed{1400 \text{ (J)}}$$

6. A tank of water is 20 ft long and has a vertical cross section in a shape of an equilateral triangle with sides 2 ft long. The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. The weight of water is 62.5 lb/ft³.



$$0 \leq y \leq \frac{18}{12} = \frac{3}{2} \text{ ft}$$

$$\text{height of the tank} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

Take a slice of water between y and $y + \Delta y$. It is a "box" of height Δy , length 20, width $2x$.

$$\text{Volume of the slice} = (2x)(20)(\Delta y)$$

Express x in terms of y .

$$\frac{x}{1} = \frac{y}{\sqrt{3}}$$

$$\text{Volume} = \frac{2x}{\sqrt{3}} (20) \Delta y$$

$$\text{weight of the slice} = (\text{volume})(62.5)$$

$$= \frac{40y}{\sqrt{3}} (62.5) \Delta y$$

distance traveled by the slice is $\sqrt{3} - y$

$$\text{work done} = \int_0^{3/2} [\text{weight}][\text{distance}] dy$$

$$= \int_0^{3/2} \frac{40y}{\sqrt{3}} (62.5) (\sqrt{3} - y) dy = \frac{40(62.5)}{\sqrt{3}} \int_0^{3/2} y(\sqrt{3} - y) dy$$

7. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

$$\begin{aligned}
 f_{\text{ave}} &= \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{\frac{\pi}{4} - (-\frac{\pi}{2})} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x dx = \frac{4}{3\pi} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x dx \quad \left| \begin{array}{l} u = \sin x \\ du = \cos x dx \\ x = -\frac{\pi}{2} \Rightarrow u = -1 \\ x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2} \end{array} \right. \\
 &= \frac{4}{3\pi} \int_{-1}^{\sqrt{2}/2} u^2 du = \frac{4}{3\pi} \left. \frac{u^3}{3} \right|_{-1}^{\sqrt{2}/2} \\
 &= \frac{4}{9\pi} \left(\frac{2\sqrt{2}}{8} - (-1) \right) = \boxed{\frac{4}{9\pi} \left(\frac{\sqrt{2}}{4} + 1 \right)}
 \end{aligned}$$

8. Evaluate the integral

$$(a) \int t^2 \cos(1 - t^3) dt \quad \left| \begin{array}{l} u = 1 - t^3 \\ du = -3t^2 dt \\ t^2 dt = -\frac{du}{3} \end{array} \right. \quad = -\frac{1}{3} \int \cos u du = \dots$$

$$(b) \int \frac{x^2}{\sqrt{1-x}} dx \quad \left| \begin{array}{l} u = 1-x \\ du = -dx \\ dx = -du \end{array} \right. \quad = - \int \frac{(1-u)^2}{\sqrt{u}} du = - \int (1-2u+u^2) u^{-1/2} du = \dots$$

$$(c) \int_0^1 x^2 e^{-x} dx = \left(x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x}) \right) \Big|_0^1 = -e^{-1} - 2e^{-1} - 2e^{-1} + 2 = \boxed{2 - 5e^{-1}}$$

Tabulation:

D	I
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

$$(d) \int x^3 e^{x^2} dx \quad \left| \begin{array}{l} u = x^2 \\ du = 2x dx \\ x dx = \frac{du}{2} \end{array} \right. = \int u e^u \frac{du}{2} = \frac{1}{2} \int u e^u du \quad \left| \begin{array}{l} u = w \\ u' = 1 \\ v = e^w \\ v' = e^w \end{array} \right.$$

$$\frac{1}{2} (u e^u - \int e^u du) = \frac{1}{2} (u e^u - e^u) + C$$

$$= \boxed{\frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + C}$$

$$(e) \int \sqrt{t} \ln t dt \quad \left| \begin{array}{l} u = \ln t \\ u' = \frac{1}{t} \\ v = \frac{t^{3/2}}{3/2} = \frac{2t^{3/2}}{3} \\ v' = \frac{3t^{1/2}}{3} = t^{1/2} \end{array} \right. = \frac{2}{3} t^{3/2} \ln t - \int \frac{1}{t} \frac{2t^{3/2}}{3} dt$$

$$= \frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \int t^{1/2} dt = \boxed{\frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \frac{t^{3/2}}{3/2} + C}$$