

Section 8.3: Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$	$1 - \sin^2 t = \cos^2 t$
$\sqrt{a^2 + x^2}$	$x = a \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$	$1 + \tan^2 t = \sec^2 t$
$\sqrt{x^2 - a^2}$	$x = a \sec t, \quad 0 \leq t \leq \frac{\pi}{2} \text{ or } \pi \leq t \leq \frac{3\pi}{2}$	$\sec^2 t - 1 = \tan^2 t$

Problems.

1. Evaluate the integral

(a) $\int \frac{x^2}{\sqrt{4+x^2}} dx$

$$(b) \int \frac{x}{\sqrt{6x - x^2}} dx$$

$$(c) \int \frac{dx}{x^2 \sqrt{25x^2 - 9}}$$

Section 8.4: Integration Of Rational Functions By Partial Fractions

- Rational function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.
- Make sure that the fraction is **proper**, that is the degree of the numerator is strictly less than the degree of denominator. Otherwise, do long divisions.
- Factor out the denominator $Q(x)$ as far as you can.
- For each factor in the denominator use the following table:

Factor	Corresponding Partial Fraction
linear factor $ax + b$	$\frac{A}{ax + b}$
linear repeated factor $(ax + b)^r$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r}$
quadratic irreducible factor $ax^2 + bx + c$, $b^2 - 4ac < 0$	$\frac{Ax + B}{ax^2 + bx + c}$
quadratic irreducible factor $(ax^2 + bx + c)^r$, $b^2 - 4ac < 0$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$

- Solve for numerators using roots of the denominator and/or matching powers of x . Then integrate.

Problems.

3. Decompose into partial fractions the rational function without computing coefficient of the decomposition:

(a) $\frac{7}{x(x^3 - 2x^2 + 2x)}$

(b) $\frac{x - 1}{(x + 2)^3(x^2 - 2x + 5)^2}$

4. Evaluate the integral

(a) $\int \frac{7}{(x - 2)(x + 5)} dx$

$$(b) \int \frac{x^5}{(x-2)^2} dx$$

$$(c) \int \frac{x^2 - 3x + 7}{(x-1)(x^2+1)} dx$$

$$(d) \int \frac{dx}{(x^2 + 1)(x^2 + x + 1)}$$

Section 8.9: Improper Integrals

- TYPE I: Infinite Interval and Continuous Integrand.

$$\begin{aligned}
 - \int_a^{\infty} f(x)dx &= \lim_{t \rightarrow \infty} \int_a^t f(x)dx \\
 - \int_{-\infty}^b f(x)dx &= \lim_{t \rightarrow -\infty} \int_t^b f(x)dx \\
 - \int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx \text{ where } a \text{ is any real number}
 \end{aligned}$$

- TYPE II: Discontinuous Integrand and Finite Interval:

$$\begin{aligned}
 - \text{ if } f \text{ is discontinuous at } b, \text{ then } \int_a^b f(x)dx &= \lim_{t \rightarrow b^-} \int_a^t f(x)dx \\
 - \text{ if } f \text{ is discontinuous at } a, \text{ then } \int_a^b f(x)dx &= \lim_{t \rightarrow a^+} \int_t^b f(x)dx \\
 - \text{ if } f \text{ has discontinuity at } c (a < c < b), \text{ then } \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx
 \end{aligned}$$

$$\bullet \boxed{\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges,} & \text{if } p \leq 1 \end{cases}} \quad \boxed{\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p}, & \text{if } p \leq 1 \\ \text{diverges,} & \text{if } p > 1 \end{cases}}$$

- **Comparison theorem.** Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is convergent.
 2. If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is divergent.
5. Compute the following integrals or show that they diverge.

(a) $\int_e^{\infty} \frac{dx}{x \ln^5 x}$

$$(b) \int_{-\infty}^0 (1+x)e^x \, dx$$

$$(c) \int_{-\infty}^{\infty} \frac{5x^4}{(x^5 + 3)^3} dx$$

$$(d) \int_0^9 \frac{dx}{\sqrt[3]{x-4}}$$

6. Determine whether the given integrals converge or diverge using the Comparison Theorem.

$$(a) \int_0^\infty \frac{dx}{x^7 + e^{7x}}$$

$$(b) \int_5^\infty \frac{x^2}{x^{5/2} - x} dx$$

$$(c) \int_{10}^\infty \frac{\sin^4(7x)}{x^7} dx$$