Math 152/172 **WEEK in REVIEW 7** **Spring 2017**

Review for Test 2 over 8.3, 8.4, 8.9, 9.3, 9.4, 10.1

1. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3+12x^2+x}{(x^3-x)(x^3+2x^2-3x)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x(x^2-1)x(x^2+2x-3)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)(x+1)(x+3)(x-1)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)^2(x+1)(x+3)(x^2+x+1)(x^2+9)^2} = \boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}+\frac{D}{(x-1)^2}+\frac{E}{x+1}+\frac{F}{x+3}+\frac{Gx+H}{x^2+x+1}+\frac{Ix+J}{x^2+9}+\frac{Kx+L}{(x^2+9)^2}}$$

2. Evaluate the integral

(a) $\int (4x^2-25)^{-3/2}\,dx$

$\sqrt{a^2-x^2}$	$x = a\sin t$
$\sqrt{x^2+a^2}$	$x = a\tan t$
$\sqrt{x^2-a^2}$	$x = a\sec t$

$2x = 5\sec t$ or $x = \frac{5}{2}\sec t$

$dx = \frac{5}{2}\sec t\tan t\,dt$

$(4x^2-25)^{1/2} = \sqrt{4x^2-25} = \sqrt{25\sec^2 t-25}$

$= \sqrt{25(\sec^2 t-1)} = \sqrt{25\tan^2 t} = 5\tan t$

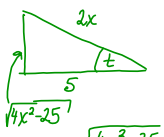
$$= \int \underbrace{\frac{1}{(5\tan t)^3}}_{\frac{1}{(\sqrt{4x^2-25})^3}} \cdot \underbrace{\frac{5}{2}\sec t\tan t\,dt}_{dx}$$

$$= \frac{5}{2}\cdot\frac{1}{125}\int \frac{\sec t\tan t\,dt}{\tan^3 t}$$

$$= \frac{1}{50}\int \frac{\sec t\,dt}{\tan^2 t} = \frac{1}{50}\int \frac{\frac{1}{\cos t}\,dt}{\frac{\sin^2 t}{\cos^2 t}}$$

$\sec t = \frac{2x}{5}$

$\cos t = \frac{5}{2x}$



$\sin t = \frac{\sqrt{4x^2-25}}{2x}$

$$= \frac{1}{50}\int \frac{1}{\cancel{\cos t}}\cdot\frac{\cos^{\cancel{2}} t}{\sin^2 t}\,dt = \frac{1}{50}\int \frac{\cos t}{\sin^2 t}\,dt \quad \left| \begin{array}{l} u = \sin t \\ du = \cos t\,dt \end{array} \right|$$

$$= \frac{1}{50}\int \frac{du}{u^2} = \frac{1}{50}\left(-\frac{1}{u}\right)+C = -\frac{1}{50\sin t}+C$$

$$= \boxed{-\frac{1}{50}\cdot\frac{2x}{\sqrt{4x^2-25}}+C}$$

(b) $\int \frac{(x-1)^2}{5\sqrt{24-x^2+2x}}dx$

complete the square:

$24-x^2+2x = 24-(x^2-2x) = 24-(x^2-2x+1-1) = 24-(x^2-2x+1)-(-1)$

$= 25-(x-1)^2$

$= \int \frac{(x-1)^2}{5\sqrt{25-(x-1)^2}}\,dx$

$x-1=5\sin t$ or $x=1+5\sin t$

$dx=5\cos t\,dt$

$\sqrt{25-(x-1)^2}=\sqrt{25-25\sin^2 t}$

$=\sqrt{25(1-\sin^2 t)}=\sqrt{25\cos^2 t}$

$=5\cos t$

$\cos t = \frac{\sqrt{25-(x-1)^2}}{5}$

$\sin t = \frac{x-1}{5}$

$t = \arcsin\left(\frac{x-1}{5}\right)$

$= \int \frac{\cancel{5}\,\cancel{25}\sin^2 t}{\cancel{5}\,\cancel{(5\cos t)}}\,\cancel{5\cos t}\,dt$

$= 5\int \sin^2 t\,dt = 5\int \frac{1-\cos 2t}{2}\,dt$

$\sin 2t = 2\sin t\cos t$

$= \frac{5}{2}\left(t-\frac{1}{2}\sin 2t\right)+C$

$= \frac{5}{2}(t-\sin t\cos t)+C$

$= \boxed{\frac{5}{2}\left(\arcsin\left(\frac{x-1}{5}\right)-\frac{(x-1)\sqrt{25-(x-1)^2}}{25}\right)+C}$

(6*) $\int \frac{x^3\,dx}{\sqrt{x^2+1}} = \int \frac{x^2\cdot x\,dx}{\sqrt{x^2+1}}$

$u=x^2+1,\ x^2=u-1$

$du=2x\,dx$

$x\,dx=\frac{du}{2}$

$\int \frac{(u-1)\frac{du}{2}}{\sqrt{u}}$

$x=\tan t$

(c) $\displaystyle\int \frac{5x^2+x+12}{x^3+4x}dx$

$$\frac{5x^2+x+12}{x^3+4x} = \frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{5x^2+x+12}{\cancel{x^3+4x}} = \frac{A(x^2+4)+x(Bx+C)}{\cancel{x(x^2+4)}}$$

$$5x^2+x+12 = Ax^2+4A+Bx^2+Cx$$

$$5x^2+x+12 = (A+B)x^2+Cx+4A$$

$$\begin{array}{l} x^2: \\ x: \\ 1: \end{array} \begin{cases} 5 = A+B \\ 1 = C \\ 12 = 4A \end{cases} \Rightarrow \boxed{A=3}$$

$$B = 5-A = \boxed{2=B}$$

$$\boxed{C=1}$$

$$\int \frac{5x^2+x+12}{x^3+4x}dx = \frac{3}{x} + \frac{2x+1}{x^2+4} = \int\left[\frac{3}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\right]dx = 3\ln|x| + \int\frac{du}{u} + \frac{1}{2}\arctan\frac{x}{2} + C$$

$u = x^2+4$

$du = 2x\,dx$

$$= 3\ln|x| + \ln|u| + \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= \boxed{3\ln|x| + \ln|x^2+4| + \frac{1}{2}\arctan\frac{x}{2} + C}$$

$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ | $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$ if $f(x)$ is discont. @ a | $\int_1^\infty \frac{dx}{x^p}$ converges, if $p>1$ | $\int_0^1 \frac{dx}{x^p}$ converges, if $p<1$

3. Determine whether the given integral is convergent or divergent.

Comparison Thm. If $f(x) \geq g(x)$ on (a,∞), then
(a) if $\int_a^\infty f(x)dx$ is conv., then $\int_a^\infty g(x)dx$ is convergent as well.
(b) if $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent as well.

(a) $\displaystyle\int_1^\infty \frac{4+\cos^4 x}{x}dx$

$0 \leq \cos^4 x \leq 1$

$\frac{4}{x} \leq \frac{4+\cos^4 x}{x} \leq \frac{5}{x}$

By Comparison Thm. $\int_1^\infty \frac{4+\cos^4 x}{x}dx$ diverge by comparison with $\int_1^\infty \frac{4}{x}dx$ $(p=1)$

(b) $\displaystyle\int_1^\infty \frac{3+\sin x}{x^2}dx$

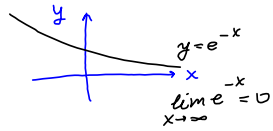
$-1 \leq \sin x \leq 1$

$-1+3 \leq 3+\sin x \leq 3+1$

$\frac{2}{x^2} \leq \frac{3+\sin x}{x^2} \leq \frac{4}{x^2}$

$\int_1^\infty \frac{3+\sin x}{x^2}dx$ converges by comparison with $\int_1^\infty \frac{4}{x^2}dx$ $(p=2>1)$

(c) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}}dx$



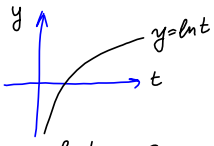
$\frac{1}{\sqrt{x}+e^{4x}} \leq \frac{1}{e^{4x}}$

$\int_0^\infty e^{-4x}dx = \lim_{t\to\infty} \int_0^t e^{-4x}dx = \lim_{t\to\infty} \left[-\frac{1}{4}e^{-4x}\right]_0^t = -\frac{1}{4}\lim_{t\to\infty}\left(\cancel{e^{-4t}}^{\,0} - e^0\right) = \frac{1}{4}$ convergent.

$\int_0^\infty \frac{dx}{\sqrt{x}+e^{4x}}$ converges by comparison with $\int_0^\infty e^{-4x}dx$

4. Compute the following integrals or show that they diverge.

(a) $\displaystyle\int_e^{\infty} \frac{dx}{x\,\ln^5 x} = \lim_{t\to\infty} \int_e^t \frac{dx}{x\,\ln^5 x}$ $\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ x = e \Rightarrow u = \ln e = 1 \\ x = t \Rightarrow u = \ln t \end{array} \right| = \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^5}$ $p = 5 > 1$ converges.

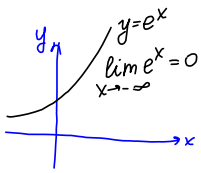


$\lim_{t\to\infty} \ln t = \infty$

$$= \lim_{t\to\infty} \frac{u^{-5+1}}{-5+1}\Big|_1^{\ln t} = \lim_{t\to\infty} \frac{u^{-4}}{-4}\Big|_1^{\ln t} = \lim_{t\to\infty} \left(-\cancel{\frac{1}{4\ln^4 t}}^{\,0} + \frac{1}{4} \right) = \boxed{\frac{1}{4}}$$

(b) $\displaystyle\int_{-\infty}^{0} (1+x)e^x\,dx = \lim_{t\to-\infty} \int_t^0 (1+x)e^x\,dx$ $\left| \begin{array}{ll} u = 1+x & v' = e^x \\ u' = 1 & v = e^x \end{array} \right|$ Integrate by parts:

$$= \lim_{t\to-\infty} \left[(1+x)e^x\Big|_t^0 - \int_t^0 e^x\,dx \right] = \lim_{t\to-\infty} \left[1 - (1+t)e^t - e^x\Big|_t^0 \right]$$



$\lim_{x\to-\infty} e^x = 0$

$$= \lim_{t\to-\infty} \left[\cancel{1} - (\cancel{1}+t)e^t - \cancel{1} + \cancel{e^t} \right]$$

$$= -\lim_{t\to-\infty} te^t \;\; |0\cdot\infty| = -\lim_{t\to-\infty} \frac{t}{e^{-t}} \overset{\text{d'Hospital's Rule}}{=\!=\!=} -\lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= +\lim_{t\to-\infty} e^t = \boxed{0} \quad \boxed{\text{converges}}$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{6x^5}{(x^6+3)^3}\,dx = \int_{-\infty}^{0} + \int_{0}^{\infty} = \lim_{t\to-\infty} \int_{t}^{0} \frac{5x^4}{(x^5+3)^3}\,dx + \lim_{s\to\infty} \int_{0}^{s} \frac{5x^4}{(x^5+3)^3}\,dx$

$$\left|\begin{array}{l} u = x^6+3 \\ du = 6x^5\,dx \\ x=t \Rightarrow u = t^6+3 \\ x=s \Rightarrow u = s^6+3 \\ x=0 \Rightarrow u=3 \end{array}\right|$$

$(p = 3 > 1)$ convergent.

$$= \lim_{t\to-\infty} \int_{t^6+3}^{3} \frac{du}{u^3} + \lim_{s\to\infty} \int_{3}^{s^6+3} \frac{du}{u^3}$$

$$= \lim_{t\to-\infty} \frac{u^{-2}}{-2}\Big|_{t^6+3}^{3} + \lim_{s\to\infty} \frac{u^{-2}}{-2}\Big|_{3}^{s^6+3}$$

$$= -\frac{1}{2(3^2)} + \frac{1}{2}\lim_{t\to-\infty} \cancelto{0}{\frac{1}{(t^6+3)^2}} - \frac{1}{2}\lim_{s\to\infty} \cancelto{0}{\frac{1}{(s^6+3)^2}} + \frac{1}{2(3^2)} = 0$$

(d) $\displaystyle\int_{0}^{2017} \frac{1}{\sqrt{2017-x}}\,dx$

discontinuity @ $x=2017$

$p = 1/2 < 1$ convergent

$$= \lim_{t\to 2017^-} \int_{0}^{t} \frac{1}{\sqrt{2017-x}}\,dx \quad \left|\begin{array}{l} u = 2017-x \\ du = -dx \\ x=0 \Rightarrow u = 2017 \\ x=t \Rightarrow u = 2017-t \end{array}\right| \quad = -\lim_{t\to 2017^-} \int_{2017}^{2017-t} u^{-1/2}\,du$$

$$= -\lim_{t\to 2017^-} \frac{u^{1/2}}{1/2}\Big|_{2017}^{2017-t} = -2\left(\cancelto{0}{\lim_{t\to 2017^-} \sqrt{2017-t}} - \sqrt{2017}\right) = \boxed{2\sqrt{2017}}$$

$S.A._x = 2\pi \int_a^b y \, ds$ | $ds = \begin{cases} \sqrt{1+[y'(x)]^2}\,dx \\ \sqrt{1+[x'(y)]^2}\,dy \\ \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt \end{cases}$ | $S.A._y = 2\pi \int_a^b x \, ds$

5. The curve $y = \sin x$ for $0 \le x \le \pi$ is rotated about the y axis. Set up, *but don't evaluate* the integral for the area of the resulting surface.

$$S.A. = 2\pi \int_0^{\pi} x \sqrt{1+[y'(x)]^2}\,dx = \boxed{2\pi \int_0^{\pi} x\sqrt{1+\cos^2 x}\,dx}$$

$y = \sin x, \quad y' = \cos x$

$y'(x) = \frac{1}{2}(e^x - e^{-x})$

6. The curve $y = \frac{1}{2}(e^x + e^{-x})$, $0 \le x \le 1$, is rotated about the x-axis. Find the area of the resulting surface.

$$S.A. = 2\pi \int_0^1 y(x)\sqrt{1+[y'(x)]^2}\,dx = \cancel{2}\pi \int_0^1 \frac{1}{\cancel{2}}(e^x+e^{-x})\sqrt{1+\left[\frac{1}{2}(e^x-e^{-x})\right]^2}\,dx$$

$$= \pi \int_0^1 (e^x+e^{-x})\sqrt{1+\underbrace{\frac{1}{4}(e^{2x} - 2\cdot e^x\cdot e^{-x} + e^{-2x})}_{\left[\frac{1}{2}(e^x-e^{-x})\right]^2}}\,dx$$

$$= \pi \int_0^1 (e^x+e^{-x})\sqrt{1+\frac{1}{4}e^{2x} - \underbrace{\frac{1}{4}\cdot 2}_{1/2} + \frac{1}{4}e^{-2x}}\,dx$$

$$= \pi \int_0^1 (e^x+e^{-x})\sqrt{\underbrace{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}_{\left[\frac{1}{2}(e^x+e^{-x})\right]^2}}\,dx = \pi \int_0^1 (e^x+e^{-x})\frac{1}{2}(e^x+e^{-x})\,dx$$

$$= \frac{\pi}{2}\int_0^1 (e^x+e^{-x})^2\,dx$$

$$= \frac{\pi}{2}\int_0^1 (e^{2x} + 2e^x e^{-x} + e^{-2x})\,dx$$

$$= \frac{\pi}{2}\int_0^1 (e^{2x} + 2 + e^{-2x})\,dx$$

$$= \frac{\pi}{2}\left(\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right)\Big|_0^1$$

$$= \boxed{\frac{\pi}{2}\left(\frac{1}{2}e^2 + \cancel{2} - \frac{1}{2}e^{-2} - \cancel{\frac{1}{2}} - \cancel{0} + \cancel{\frac{1}{2}}\right)}$$

$$= \boxed{\frac{\pi}{4}(e^2 - e^{-2})}$$

$L = \int_a^b ds$

$x'(t) = 4t,\ y'(t) = 3t^2$

7. Set up, *but don't evaluate* the integral for the length of the curve $x = 2t^2$, $y = t^3$, $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \boxed{\int_0^1 \sqrt{16t^2 + 9t^4}\, dt}$$

$0 \le x \le \frac{1}{6}$

8. Find length of the curve $y = \frac{1}{\pi}\ln(\sec(\pi x))$ from the point $(0,0)$ to the point $\left(\frac{1}{6}, \ln\frac{2}{\sqrt{3}}\right)$.

$$y'(x) = \frac{1}{\cancel{\pi}} \frac{\cancel{\sec(\pi x)}\tan(\pi x)\cancel{(\pi)}}{\cancel{\sec(\pi x)}}$$

$$y'(x) = \tan(\pi x)$$

$$L = \int_0^{1/6} \sqrt{1 + [y'(x)]^2}\, dx = \int_0^{1/6} \sqrt{1 + \tan^2(\pi x)}\, dx = \int_0^{1/6} \sqrt{\sec^2(\pi x)}\, dx$$

$$= \int_0^{1/6} \sec(\pi x)\, dx = \frac{1}{\pi}\left[\ln|\sec \pi x + \tan \pi x|\right]_0^{1/6}$$

$$= \frac{1}{\pi}\left(-\ln|\cancelto{1}{\sec 0} + \cancelto{0}{\tan 0}| + \ln|\cancelto{\frac{2}{\sqrt{3}}}{\sec\frac{\pi}{6}} + \cancelto{\frac{1}{\sqrt{3}}}{\tan\frac{\pi}{6}}|\right)$$

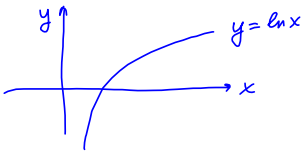
$$= \frac{1}{\pi}\left(\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln 1\right) = \frac{1}{\pi}\ln(\sqrt{3}) = \boxed{\frac{1}{2\pi}\ln 3}$$

9. Determine if the sequence $\{a_n\}_{n=2}^{\infty}$ is decreasing and bounded:

(a) $a_n = \ln n$

increasing

non bounded



- increasing
- $\lim_{x\to\infty} \ln x = \infty$

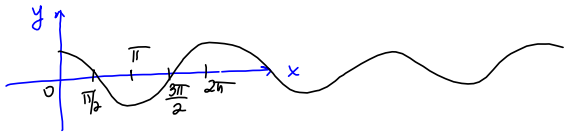
$\lim_{x\to 0^+} \ln x = -\infty$

non bounded

(b) $a_n = \cos(n^2)$

$-1 \le \cos(n^2) \le 1$ bounded

non monotonic

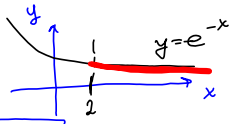


(c) $a_n = e^{-n}$

if $n > 2$, then

$0 \le e^{-n} \le e^{-2}$ - $\boxed{\text{bounded}}$

$\boxed{\text{decreasing}}$

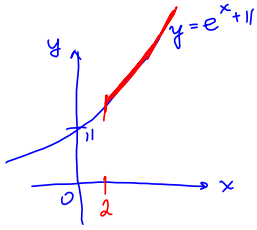


, $x \ge 2$

(d) $a_n = e^n + 11$, $n \ge 2$

increasing

non bounded



(e) $a_n = 1 - \frac{1}{n^2}$

$0 < a_n < 1$ bounded.

$f(x) = 1 - \frac{1}{x^2}$, $f'(x) = \frac{2}{x^3} > 0$ on $[2, \infty)$

increasing.

10. Determine if the sequence converges or diverges. If converges, find its limit.

(a) $\left\{\dfrac{2017+(-1)^n}{n^{2017}}\right\}_{n=1}^{\infty}$

$a_n = \dfrac{2017+(-1)^n}{n^{2017}}$

if n is even, then $(-1)^n = 1$, $a_n = \dfrac{2017+1}{n^{2017}}$, $\lim\limits_{n\to\infty} \dfrac{2018}{n^{2017}} = 0$

match

if n is odd, then $(-1)^n = -1$, $a_n = \dfrac{2017-1}{n^{2017}}$, $\lim\limits_{n\to\infty} \dfrac{2016}{n^{2017}} = 0$

$\lim\limits_{n\to\infty} a_n = \boxed{0}$ convergent.

(b) $\left\{\dfrac{7n+6n^3+n^2}{(n+3)(n^2+8)}\right\}_{n=4}^{\infty}$

$\lim\limits_{n\to\infty} \dfrac{7n+6\boxed{n^3}+n^2}{(n+3)(n^2+8)} = \boxed{6}$ convergent.

$a_n > 0$

11. Assuming that the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{16}{a_n}\right)$ is convergent, find its limit.

$L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$

$(2L)\ L = \frac{1}{2}\left(L + \frac{16}{L}\right)(2L)$

$2L^2 = (L^2+16)$

$L^2 = 16 \Rightarrow \boxed{L=4}$ or $L=-4$.