Sections 10.2, 10.3, 10.4.

10.2: Series

- Infinite series $\sum_{n=1}^{\infty} a_n$ ($n = 1$ for convenience, it can be anything).
- Partial sums: $s_N = \sum_{n=1}^{N} a_n$. Note $s_N = s_{N-1} + a_N$.
- If $\{s_N\}_{N=1}^{\infty}$ is convergent and $\lim_{N\to\infty} s_N = s$ exists as a real number, then the series $\sum_{n=1}^{n} a_n$ is *convergent*. The number s is called the **sum** of the series.
- Series we can sum:
  - Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad -1 < r < 1$
  - Telescoping Series
- THE TEST FOR DIVERGENCE: *If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.*
- **The Test for Divergence cannot be used to prove that a series converges. It can only show a series is divergent.**

Examples

1. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.
2. Find the sum of the following series or show they are divergent:
   (a) $\sum_{n=1}^{\infty} \frac{7+5^n}{10^n}$
   (b) $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)}$
3. Write the repeating decimal $0.\overline{27}$ as a fraction.
4. Use the test for Divergence to determine whether the series diverges.
   (a) $\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$
   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6 - e^{-n}}$

10.3: The Integral and Comparison Tests; Estimating Sums

- THE TEST FOR DIVERGENCE: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.
- THE INTEGRAL TEST: Let $\sum a_n$ be a **positive** series. If f is a continuous and decreasing function on $[a, \infty)$ such that $a_n = f(n)$ for all $n \geq a$ then $\sum a_n$ and $\displaystyle\int_a^{\infty} f(x)\,\mathrm{d}x$ both converge or both diverge.
- THE COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **nonnegative** terms and $a_n \leq b_n$ for all n.
  1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
  2. If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.
- LIMIT COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms . If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.
- The p-series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.
- REMAINDER ESTIMATE FOR THE INTEGRAL TEST: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_n^{\infty} f(x)\,\mathrm{d}x$$

Examples.

6. (a) If $\displaystyle\sum_{n=1}^{1000} \frac{1}{n^6}$ is used to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

   (b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

7. Given the series $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n^4}$.

   (a) Show that the series converges.

   (b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

8. Find the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

9. Determine if the following series is convergent or divergent:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5}\,\sqrt[3]{8n}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+12}{\sqrt{n^6+6}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^5+e^{-5n}}{6n^6-e^{-6n}}$

10. Find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ is convergent.

10.4 : Other Convergence Tests

- ALTERNATING SERIES TEST: If $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum(-1)^n b_n$ is convergent.

- RATIO TEST: For a series $\sum a_n$ with nonzero terms define $L = \displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

  1. If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
  2. If $L > 1$ then the series is divergent.
  3. If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

- **The Alternating Series Theorem.** If $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then $|R_n| \le b_{n+1}$.

Examples

12. Which of the following statements is TRUE?

   (a) If $a_n > 0$ for $n \ge 1$ and $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $a_n > 0$ for $n \ge 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(c) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(d) If $a_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

13. Determine whether the following series converges absolutely, converges but not absolutely, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$, where p is a real parameter.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}$

(c) $\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)!}$

(d) $\sum_{n=5}^{\infty} \frac{(-1)^{n-1}7^{n-1}}{4^n}$

(e) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+n+1}$

14. Given the series $\sum_{n=1}^{\infty}(-1)^{n+1}n^3 e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.