Math 152/172 **WEEK in REVIEW 8** **Spring 2017**

Sections 10.2, 10.3, 10.4.

10.2: Series

- Infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ ($n = 1$ for convenience, it can be anything).
- Partial sums: $s_N = \displaystyle\sum_{n=1}^{N} a_n$. Note $s_N = s_{N-1} + a_N$.
- If $\{s_N\}_{N=1}^{\infty}$ is convergent and $\lim\limits_{N\to\infty} s_N = s$ exists as a real number, then the series $\displaystyle\sum_{n=1}^{n} a_n$ is *convergent*. The number s is called the **sum** of the series.
- Series we can sum:
  - Geometric Series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $-1 < r < 1$
  - Telescoping Series
- THE TEST FOR DIVERGENCE: *If* $\lim\limits_{n\to\infty} a_n$ *does not exist or if* $\lim\limits_{n\to\infty} a_n \neq 0$, *then the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is divergent.*
- **The Test for Divergence cannot be used to prove that a series converges. It can only show a series is divergent.**

Examples

1. Given a series whose partial sums are given by $s_n = (7n+3)/(n+7)$, find the general term a_n of the series and determine if the series converges or diverges. If it converges, find the sum.

$S_n = \frac{7n+3}{n+7}$ - partial sum.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{7n+3}{n+7} = \lim_{n\to\infty} \frac{\cancel{n}(7+\overset{0}{3/n})}{\cancel{n}(1+\underset{0}{7/n})} = \boxed{7 = \text{sum of the series}}$$

$$S_n = \underbrace{a_1 + a_2 + \dots + a_{n-1}}_{S_{n-1}} + a_n \Rightarrow \boxed{a_n = S_n - S_{n-1}}$$

$$S_{n-1} = \frac{7(n-1)+3}{(n-1)+7} = \frac{7n-7+3}{n-1+7} = \frac{7n-4}{n+6}$$

$$a_n = S_n - S_{n-1} = \frac{7n+3}{n+7} - \frac{7n-4}{n+6} = \frac{(7n+3)(n+6) - (7n-4)(n+7)}{(n+7)(n+6)}$$

$$= \frac{\cancel{7n^2} + \underbrace{42n + 3n}_{45n} + 18 - (\cancel{7n^2} + \underbrace{49n - 4n}_{45n} - 28)}{(n+7)(n+6)} = \frac{18+28}{(n+7)(n+6)} = \boxed{\frac{46}{(n+7)(n+6)}}$$

$r = \frac{1}{10} < 1$ | geometric, $r = \frac{1}{2} < 1$ converge

2. Find the sum of the following series or show they are divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7+5^n}{10^n}$ $= \displaystyle\sum_{n=1}^{\infty} \frac{7}{10^n} + \sum_{n=1}^{\infty} \frac{5^n}{10^n} = 7\sum_{n=1}^{\infty} \frac{1}{10^n} + \sum_{n=1}^{\infty} \left(\frac{5}{10}\right)^n = 7\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$= 7\displaystyle\sum_{n=1}^{\infty} \frac{1}{10}\left(\frac{1}{10}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}$

$= \frac{7}{10}\,\frac{1}{1-1/10} + \frac{1}{2}\cdot\frac{1}{1-1/2} = \cancel{\frac{7}{10}}\cdot\cancel{\frac{10}{9}} + \cancel{\frac{1}{2}}\,\cancel{\frac{1}{1/2}} = \frac{7}{9} + 1 = \boxed{\frac{16}{9}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+3)}$

Partial fractions: $\dfrac{8}{(n+1)(n+3)} = \dfrac{A}{n+1} + \dfrac{B}{n+3}$

$8 = A(n+3) + B(n+1)$

$n=-3$: $8 = -2B \Rightarrow B = -4$

$n=-1$: $8 = 2A \Rightarrow A = 4$

$\dfrac{8}{(n+1)(n+3)} = 4\left[\dfrac{1}{n+1} - \dfrac{1}{n+3}\right] = a_n$

Partial sums: $a_1 = S_1 = 4\left(\frac{1}{2} - \frac{1}{4}\right)$

$S_2 = S_1 + a_2 = 4\left(\frac{1}{2} - \frac{1}{4}\right) + 4\left(\frac{1}{3} - \frac{1}{5}\right) = 4\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5}\right)$

$S_3 = S_2 + a_3 = 4\left(\frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{4}} - \frac{1}{5}\right) + 4\left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$

$= 4\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6}\right)$

$$\boxed{S_n = 4\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right)}$$

Sum of the series: $S = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} 4\left(\frac{1}{2} + \frac{1}{3} - \cancelto{0}{\frac{1}{n+2}} - \cancelto{0}{\frac{1}{n+3}}\right) = 4\left(\frac{1}{2} + \frac{1}{3}\right) = \boxed{\frac{10}{3}}$

3. Write the repeating decimal $0.\overline{27}$ as a fraction.

$$0.\overline{27} = 0.\underbrace{27}\underbrace{27}\underbrace{27}27\ldots = 0.27 + 0.0027 + 0.000027 + \ldots$$

$$= \frac{27}{100} + \frac{27}{10000} + \frac{27}{10^6} + \ldots = \frac{27}{100}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right]$$

$$= \frac{27}{100}\sum_{n=0}^{\infty}\left(\frac{1}{10^2}\right)^n = \frac{27}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n \leftarrow \text{geometric } r = \frac{1}{100}$$

$$= \frac{27}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{27}{\cancel{100}} \cdot \frac{1}{\frac{99}{\cancel{100}}} = \boxed{\frac{27}{99}}$$

4. Use the test for Divergence to determine whether the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{3(n^4+3)(n+1)}$

$a_n = \dfrac{n^5}{3(n^4+3)(n+1)}$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^5}{3(n^4+3)(n+1)} = \lim_{n\to\infty} \frac{n^5}{3n^4(1+\frac{3}{n^4})n(1+\frac{1}{n})}$

$\displaystyle = \lim_{n\to\infty} \frac{\cancel{n^5}}{3\cancel{n^4 n}} = \frac{1}{3} \neq 0$

$\boxed{\text{Divergent by Divergence Test}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

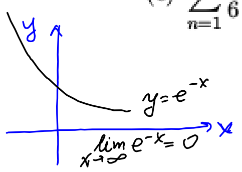
$a_n = \dfrac{(-1)^n}{n\sqrt{n}}$, $|a_n| = \left|\dfrac{(-1)^n}{n\sqrt{n}}\right| = \dfrac{1}{n\sqrt{n}}$

$\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0.$

$\boxed{\text{Thm } \lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0 \text{ (section 10.1).}}$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n\sqrt{n}} = 0$ $\boxed{\text{DIVERGENCE TEST FAILS.}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6-e^{-n}}$



$\displaystyle\lim_{n\to\infty} \frac{1}{6-\cancel{e^{-n}}^{\,0}} = \frac{1}{6} \neq 0$

$\boxed{\text{divergent by the Divergence Test}}$

10.3: The Integral and Comparison Tests; Estimating Sums

(D.T.)

- THE TEST FOR DIVERGENCE: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

(I.T.)

- THE INTEGRAL TEST: Let $\sum a_n$ be a **positive** series. If f is a continuous and decreasing function on $[a, \infty)$ such that $a_n = f(n)$ for all $n \geq a$ then $\sum a_n$ and $\int_a^\infty f(x)\,\mathrm{d}x$ both converge or both diverge.

(C.T.)

- THE COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **nonnegative** terms and $a_n \leq b_n$ for all n.
  1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
  2. If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.

(L.C.T.)

- LIMIT COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms . If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where c is a finite number and $c > 0$, then either both series converge or both diverge.

- The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.

- REMAINDER ESTIMATE FOR THE INTEGRAL TEST: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then
$$\boxed{\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_n^{\infty} f(x)\,\mathrm{d}x}$$

Examples.

$$\sum_{n=1}^{\infty} \frac{1}{n^6} \approx \underbrace{\frac{1}{1^6} + \frac{1}{2^6} + \ldots + \frac{1}{1000^6}}_{1000}$$

6. (a) If $\overbrace{\sum_{n=1}^{1000} \frac{1}{n^6}}^{S_{1000}}$ is used to approximate $\sum_{n=1}^{\infty} \frac{1}{n^6}$, find an upper bound on the error using the Integral Test.

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ correct to 11 decimal places.

(a) $\int_{1001}^{\infty} f(x)dx \leq R_{1000} \leq \int_{1000}^{\infty} f(x)dx$, $f(x) = \frac{1}{x^6}$

$\int_a^{\infty} \frac{1}{x^6} dx = \lim_{t\to\infty} \int_a^t x^{-6} dx$

$= \lim_{t\to\infty} \frac{x^{-5}}{-5}\Big|_a^t = \lim_{t\to\infty} \left(-\frac{1}{5x^5}\right)\Big|_a^t$

$= \lim_{t\to\infty} \left(-\cancel{\frac{1}{5t^5}}^{\,0} + \frac{1}{5a^5}\right) = \frac{1}{5a^5}$

$$\boxed{\frac{1}{5(1001)^5} \leq R_{1000} \leq \frac{1}{5\cdot(1000)^5}}$$

(b) $R_n < 10^{-11}$

$R_n \leq \int_n^{\infty} \frac{1}{x^6} dx = \frac{1}{5n^5} < 10^{-11}$

$\frac{1}{5n^5} < \frac{1}{10^{11}}$

$5n^5 > 10^{11}$

$n^5 > \frac{10^{11}}{5}$

$n > \sqrt[5]{\frac{10^{11}}{5}} \approx 114.8698$, $n = 115$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} \approx \frac{1}{1^6} + \frac{1}{2^6} + \ldots + \frac{1}{115^6}$$

7. Given the series $\sum_{n=1}^{\infty} n^3 e^{-n^4}$.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

(a) Integral Test for $f(x) = x^3 e^{-x^4}$

Need to show that $f(x)$ is continuous, decreasing, positive on $[1,\infty)$

- $f(x) = x^3 e^{-x^4}$, since $x \geq 1$ and $e^{-x^4} > 0$, then $x^3 e^{-x^4} > 0$ on $[1,\infty)$
- both x^3 and e^{-x^4} are continuous on $(-\infty,\infty)$, thus $x^3 e^{-x^4}$ is continuous on $(-\infty, \infty)$.
- $f'(x) = 3x^2 e^{-x^4} + x^3 \cdot e^{-x^4}(-4x^3) = 3x^2 e^{-x^4} - 4x^6 e^{-x^4} = \overbrace{x^2}^{\geq 0}(3-x^4)\underbrace{e^{-x^4}}_{>0}$

$3 - x^4 < 0$ for $x \geq 2$

$f(x)$ is decreasing on $[2,\infty)$

$$\int_1^\infty x^3 e^{-x^4}\,dx = \lim_{t\to\infty} \int_1^t x^3 e^{-x^4}\,dx$$

$u = -x^4$
$du = -4x^3\,dx$
$x = 1 \Rightarrow u = -1$
$x = t \Rightarrow u = -t^4$

$$= -\frac{1}{4} \lim_{t\to\infty} \int_{-1}^{-t^4} e^u\,du$$

$$= -\frac{1}{4}\Big[\lim_{t\to\infty} \overbrace{e^{-t^4}}^{0} - e^{-1}\Big] = \frac{e^{-1}}{4}$$

the integral converges.

$$\boxed{\sum_{n=1}^{\infty} n^3 e^{-n^4} \text{ is convergent by the Integral test}}$$

(b) $\sum_{n=1}^{\infty} n^3 e^{-n^4} \approx S_5$

$$R_5 \leq \underbrace{\int_5^\infty x^3 e^{-x^4}\,dx}_{\frac{e^{-625}}{4}}$$

$u = -x^4$
$x = 5 \Rightarrow u = -5^4 = -625$

$$\boxed{R_5 \leq \frac{e^{-625}}{4}}$$

$p > 0$

8. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

- $f(x) = \frac{1}{x(\ln x)^p}$ when $x \geq 2$, then $\ln x > 0$. , $f(x) > 0$ on $[2, \infty)$
- $f(x)$ has discontinuities @ $x = 0$ and $\ln x = 0$ or $x = 1$
  continuous on $[2, \infty)$.
- show that $x(\ln x)^p$ is increasing on $[2, \infty)$

$$(x(\ln x)^p)' = (\ln x)^p + x p (\ln x)^{p-1} \cdot \frac{1}{x}$$
$$= (\ln x)^{p-1}(\ln x + p) > 0 \quad \text{on } [2, \infty),$$

so $\frac{1}{x(\ln x)^p}$ is decreasing on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx \quad \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \\ \lim_{x \to \infty} u = \infty \\ x = 2 \Rightarrow u = \ln 2 \end{array} \right| = \int_{\ln 2}^{\infty} \frac{du}{u^p} \begin{cases} \text{convergent, if } p > 1 \\ \text{divergent, if } p \leq 1 \end{cases}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad \text{converges, if } p > 1, \quad \boxed{\text{diverges, if } p \leq 1}$$

9. Determine if the following series is convergent or divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2012}{\sqrt[7]{n^5}\sqrt[3]{8n}}$ $= 2012 \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/7} \cdot 2 \cdot n^{1/3}} = \frac{2012}{2} \sum_{n=1}^{\infty} \frac{1}{n^{5/7+1/3}} = \frac{2012}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{22}{21}}}$

p-series, $p = \frac{22}{21} > 1$

$\boxed{\text{convergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+12}{\sqrt{n^6+6}}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6}} = \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series divergent.

do the limit comparison test for $a_n = \frac{n^2+12}{\sqrt{n^6+6}}$ and $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+12}{\sqrt{n^6+6}} \cdot n = \lim_{n\to\infty} \frac{n \cdot n^2\left(1+\frac{12}{n^2}\right)}{\sqrt{n^6\left(1+\frac{6}{n^6}\right)}} = \lim_{n\to\infty} \frac{n^3}{\sqrt{n^6}} = 1 > 0$$

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2+12}{\sqrt{n^6+6}} \text{ is divergent by L.C.T.}}$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \sin x \le x$ near zero.

(c) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right)$

$\sin\left(\frac{1}{n^7}\right) \le \frac{1}{n^7}$

$\sum_{n=1}^{\infty} \frac{1}{n^7}$ p-series $p = 7 > 1$ convergent.

$\boxed{\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^7}\right) \text{ converges by C.T.}}$

(d) $\sum_{n=1}^{\infty} \frac{5n^5 + e^{-5n}}{6n^6 - e^{-6n}}$

$\lim_{x\to\infty} e^{-x} = 0$

$\frac{5n^5 + e^{-5n}}{6n^6 - e^{-6n}} > \frac{5n^5}{6n^6} = \frac{5}{6n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$\boxed{\sum_{n=1}^{\infty} \frac{5n^5 + e^{-5n}}{6n^6 - e^{-6n}} \text{ is divergent by C.T.}}$

10. Find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty} \overbrace{\frac{1}{(n+1)n^p}}^{a_n}$ is convergent.

compare with $\displaystyle\frac{1}{n \cdot n^p} = \underbrace{\frac{1}{n^{p+1}}}_{b_n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)n^p}}{\frac{1}{n^{p+1}}} = \lim_{n\to\infty} \frac{n^{\cancel{p+1}}}{(n+1)\cancel{n^p}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)n^p}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ either both converge or both diverge. by L.C.T.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ is convergent if $p+1>1$ or $\boxed{p>0}$

10.4 : Other Convergence Tests

(A.S.T.)

- ALTERNATING SERIES TEST: If $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum(-1)^n b_n$ is convergent.

(R.T.)

- RATIO TEST: For a series $\sum a_n$ with nonzero terms define $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

  1. If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
  2. If $L > 1$ then the series is divergent.
  3. If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

- **The Alternating Series Theorem.** If $\sum_{n=1}^{\infty}(-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then $|R_n| \le b_{n+1}$.

Examples

12. Which of the following statements is TRUE?

(a) If $a_n > 0$ for $n \ge 1$ and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. $\boxed{\text{not true}}$.

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is convergent, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

(b) If $a_n > 0$ for $n \ge 1$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. $\boxed{\text{TRUE}}$

$|(-1)^n a_n| = a_n$

$\sum_{n=1}^{\infty} a_n$ convergent means that $\sum_{n=1}^{\infty}(-1)^n a_n$ is absolutely convergent $\Rightarrow$ convergent.

(c) If $\lim_{n\to\infty} a_n = 0$ & $a_{n+1} \le a_n$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. not true.

(d) If $a_n > 0$ for $n \ge 1$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{e}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges. not true.

$\frac{e}{2} > 1$, the series is divergent by R.T.

13. Determine whether the following series converges absolutely, $\underbrace{\text{converges but not absolutely}}_{\text{conditionally converges}}$, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$, where p is a real parameter.

$\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}$ - convergent if $p>1$, divergent, if $p\le 1$.

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p} \text{ is absolutely convergent, if } p>1}$$

$0<p\le 1$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ alternating series for $b_n = \frac{1}{n^p}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^p} = 0$$

$$b_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = b_n$$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ $(0<p\le 1)$ is convergent by A.S.T.

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p} \text{ is convergent, but not absolutely convergent, if } 0<p\le 1}$$

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}$ alternating, for $b_n = \frac{1}{n\sqrt[4]{\ln n}}$

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\sqrt[4]{\ln n}} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/4}}$ $p = 1/4 < 1$ divergent. not absolutely convergent.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\sqrt[4]{\ln n}} = 0$.

function $\frac{1}{n\sqrt[4]{\ln n}}$ is decreasing

so $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[4]{\ln n}}$ is convergent by A.S.T.

$\boxed{\text{convergent, but not absolutely convergent}}$

(c) $\sum_{n=1}^{\infty} \frac{(-9)^n}{(n+1)!}$

Ratio Test for $a_n = \frac{(-9)^n}{(n+1)!}$, $a_{n+1} = \frac{(-9)^{n+1}}{(n+2)!} = \frac{(-9)^n(-9)}{(n+2)(n+1)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-9)^n}(-9)}{(n+2)\cancel{(n+1)!}}}{\frac{\cancel{(-9)^n}}{\cancel{(n+1)!}}} \right| = \lim_{n\to\infty} \left| \frac{-9}{n+2} \right| = \lim_{n\to\infty} \frac{9}{n+2} = 0 < 1$$

$\boxed{\text{absolutely convergent by R.T.}}$

(d) $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^{n-1}7^{n-1}}{4^n}$

Ratio Test for $a_n = \dfrac{\overbrace{(-1)^{n-1}\,7^{n-1}}^{(-7)^{n-1}}}{4^n}$, $a_{n+1} = \dfrac{(-1)^n \cdot 7^n}{4^{n+1}} = \dfrac{(-7)\cdot(-7)^{n-1}}{4 \cdot 4^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-7)\cdot\cancel{(-7)^{n-1}}}{4\cdot\cancel{4^n}}}{\dfrac{\cancel{(-7)^{n-1}}}{\cancel{4^n}}} \right| = \lim_{n\to\infty} \left| -\frac{7}{4} \right| = \frac{7}{4} > 1 \quad \boxed{\text{divergent by R.T.}}$$

(e) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+n+1}$ alternating for $b_n = \frac{n}{n^2+n+1}$

$\cos \pi n = (-1)^n$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+n+1} \right| = \sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+n+1}}_{a_n}$ compare with $\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{b_n}$ divergent.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 > 0$

$\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$ is divergent by L.C.T.

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+n+1}$ - convergent by A.S.T.

$$\begin{cases} \lim_{n\to\infty} \frac{n}{n^2+n+1} = 0 \\ \left(\frac{x}{x^2+x+1}\right)' = \frac{1-x^2}{(x^2+x+1)^2} \le 0 \quad \text{on } [1,\infty) \end{cases}$$

$\boxed{\text{convergent, but not absolutely}}$

14. Given the series $\sum_{n=1}^{\infty}(-1)^{n+1}n^3e^{-n^4}$. , $b_n = n^3e^{-n^4}$ alternating series.

(a) Show that the series converges.

(b) Find an upper bound for the error approximating this series by its 5th partial sum s_5.

(a) showed that $\sum_{n=1}^{\infty} |(-1)^{n+1} n^3 e^{-n^4}| = \sum_{n=1}^{\infty} n^3 e^{-n^4}$ is convergent.

$\sum_{n=1}^{\infty} (-1)^{n+1} n^3 e^{-n^4}$ is absolutely converges $\Rightarrow$ convergent.

(b) $R_5 \leq b_{5+1} = b_6 = 6^3 e^{-6^4}$

$$\boxed{R_5 \leq 216 e^{-1296}}$$