Math 152/172 **WEEK in REVIEW 9** **Spring 2017**

Sections 10.5, 10.6, 10.7

10.5: Power Series

- For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:
  1. There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the **radius of convergence.**
  2. The series converges only for $x = a$ (then $R = 0$).
  3. The series converges for all x (then $R = \infty$).
- We find the radius of convergence using the **Ratio Test.**
- An **interval of convergence** is the interval of all x's for which the power series converges.
- You must check the endpoints $x = a \pm R$ individually to determine whether or not they are in the interval of convergence.

1. For the following series find the radius and interval of convergence.

(a) $\sum_{n=0}^{\infty} \frac{n^4 x^n}{7^n}$, $a_n = \frac{n^4 x^n}{7^n}$, $a_{n+1} = \frac{(n+1)^4 x^{n+1}}{7^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^4 x^{n+1}}{7^{n+1}}}{\frac{n^4 x^n}{7^n}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^4 x^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n^4 x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^4 \cdot x}{7n^4}\right|$$

$$= \frac{|x|}{7} \lim_{n\to\infty} \frac{(n+1)^4}{n^4} = \frac{|x|}{7} < 1$$

or $|x| < 7$

$-7 < x < 7$

$x = -7$: $\sum_{n=0}^{\infty} \frac{n^4 \overbrace{(-7)}^{(-1)\cdot 7}{}^n}{7^n} = \sum_{n=0}^{\infty} \frac{n^4 (-1)^n \cancel{7^n}}{\cancel{7^n}} = \sum_{n=0}^{\infty} (-1)^n n^4$ - alternating, $b_n = n^4$

$\lim_{n\to\infty} n^4 = \infty$, divergent by A.S.T.

$x = 7$: $\sum_{n=0}^{\infty} \frac{n^4 \cancel{7^n}}{\cancel{7^n}} = \sum_{n=0}^{\infty} n^4$

$\lim_{n\to\infty} n^4 = \infty$, divergent by D.T.

$$\boxed{\begin{array}{l} \text{interval of convergence } (-7, 7) \\ \text{radius of convergence } R = \frac{7-(-7)}{2} = 7 \end{array}}$$

(b) $\sum_{n=0}^{\infty} \frac{8^n (x+4)^{3n}}{n^3+1}$, $a_n = \frac{8^n (x+4)^{3n}}{n^3+1}$, $a_{n+1} = \frac{8^{n+1}(x+4)^{3(n+1)}}{(n+1)^3+1} = \frac{8^{n+1}(x+4)^{3n+3}}{(n+1)^3+1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{8^{n+1}(x+4)^{3n+3}}{(n+1)^3+1}}{\frac{8^n (x+4)^{3n}}{n^3+1}} \right| = \lim_{n\to\infty} \left| \frac{8(x+4)^3 (n^3+1)}{(n+1)^3+1} \right| = 8|x+4|^3 \lim_{n\to\infty} \frac{n^3+1}{(n+1)^3+1} \quad 1$$

$$= 8|x+4|^3 < 1$$

$$|x+4|^3 < \frac{1}{8} \quad \text{or} \quad |x+4| < \frac{1}{2}$$

$$-\frac{1}{2} < x+4 < \frac{1}{2}$$

$$-\frac{1}{2} - 4 < x < \frac{1}{2} - 4$$

$$-\frac{9}{2} < x < -\frac{7}{2}$$

$x = -\frac{7}{2}$: $\sum_{n=0}^{\infty} \frac{8^n \left(-\frac{7}{2}+4\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{8^n \left(\frac{1}{2}\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{8^n \left[\left(\frac{1}{2}\right)^3\right]^n}{n^3+1} = \sum_{n=0}^{\infty} \frac{1}{n^3+1}$

$\frac{1}{n^3+1} < \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is conv. (p-series, $p = 3 > 1$)

$\sum_{n=0}^{\infty} \frac{1}{n^3+1}$ converges by C.T.

$x = -\frac{9}{2}$: $\sum_{n=0}^{\infty} \frac{8^n \left(-\frac{9}{2}+4\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{8^n \left(-\frac{1}{2}\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{8^n (-1)^{3n} \left(\frac{1}{2}\right)^{3n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{8^n (-1)^{3n} \left(\frac{1}{8}\right)^n}{n^3+1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{3n}}{n^3+1}$ alternating, $b_n = \frac{1}{n^3+1}$

- $\lim_{n\to\infty} \frac{1}{n^3+1} = 0$
- $(n+1)^3+1 > n^3+1 \Rightarrow \frac{1}{(n+1)^3+1} < \frac{1}{n^3+1} \Rightarrow b_{n+1} < b_n$.

convergent by A.S.T.

$$\boxed{\begin{array}{ll} \text{interval of convergence} & \left[-\frac{9}{2}, -\frac{7}{2}\right] \\ \text{radius of convergence} & R = \frac{-7/2 - (-9/2)}{2} = \frac{1}{2} \end{array}}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n(5x-3)^n}{n}$, $a_n = \dfrac{(-9)^n(5x-3)^n}{n}$, $a_{n+1} = \dfrac{(-9)^{n+1}(5x-3)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-9)^{\cancel{n}+1}(5x-3)^{\cancel{n}+1}}{n+1}}{\dfrac{\cancel{(-9)^n}\cancel{(5x-3)^n}}{n}}\right| = \lim_{n\to\infty}\left|\frac{-9(5x-3)\,n}{n+1}\right| = |-9(5x-3)|\cancel{\lim_{n\to\infty}\frac{n}{n+1}}^{\,1}$$

$$= |-9(5x-3)| = 9|5x-3| < 1$$

$$|5x-3| < \frac{1}{9} \quad \text{or} \quad -\frac{1}{9} < 5x-3 < \frac{1}{9}$$

$$\frac{26}{9} < 5x < \frac{28}{9}$$

$$\underbrace{\frac{26}{45} < x < \frac{28}{45}}$$

$x = \dfrac{28}{45}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n\left(\frac{28}{45}\cdot 5-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\left(\frac{28}{9}-3\right)^n}{n}$

$$= \sum_{n=1}^{\infty}\frac{\cancel{(-9)^n}\cdot\cancel{\left(\frac{1}{9}\right)^n}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \quad \text{alternating for } b_n = \frac{1}{n}$$

- $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$
- $n+1 > n \Rightarrow \dfrac{1}{n+1} < \dfrac{1}{n} \Rightarrow b_{n+1} < b_n$

Convergent by A.S.T.

$x = \dfrac{26}{45}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-9)^n\left(\frac{26}{45}\cdot 5-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-9)^n\cdot\left(\frac{26}{9}-3\right)^n}{n} = \sum_{n=1}^{\infty}\frac{\cancel{(-9)^n}\cancel{\left(-\frac{1}{9}\right)^n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

harmonic, divergent

$$\boxed{\begin{array}{l}\text{interval of convergence is } \left(\frac{26}{45}, \frac{28}{45}\right] \\ \text{radius of convergence } R = \dfrac{\frac{28}{45}-\frac{26}{45}}{2} = \dfrac{1}{45}\end{array}}$$

(d) $\sum_{n=1}^{\infty} \frac{(n+1)!(x-1)^{n+1}}{4^{n+1}}$, $a_n = \frac{(n+1)!\,(x-1)^{n+1}}{4^{n+1}}$, $a_{n+1} = \frac{(n+2)!\,(x-1)^{n+2}}{4^{n+2}}$

$(n+2)! = (n+2)(n+1)!$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+2)!\,(x-1)^{n+2}}{4^{n+2}}}{\frac{(n+1)!\,(x-1)^{n+1}}{4^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!\,(x-1)}{4\,(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\cancel{(n+1)!}\,(x-1)}{4\cancel{(n+1)!}}\right|$$

$$= \frac{|x-1|}{4}\lim_{n\to\infty}(n+2) = \infty > 1 \text{ for all } x.$$

$$\boxed{\text{converges @ } x=1 \text{ only} \quad R=0}$$

(e) $\sum_{n=0}^{\infty} \frac{(-6)^n x^n}{(3n+1)!}$, $a_n = \frac{(-6)^n x^n}{(3n+1)!}$, $a_{n+1} = \frac{(-6)^{n+1} x^{n+1}}{(3(n+1)+1)!} = \frac{(-6)^{n+1} x^{n+1}}{(3n+4)!}$

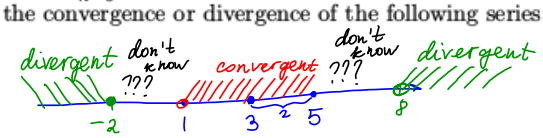
$(3n+4)! = (3n+4)(3n+3)(3n+2)(3n+1)!$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-6)^{\cancel{n+1}} x^{\cancel{n+1}}}{(3n+4)!}}{\frac{\cancel{(-6)^n} \cancel{x^n}}{(3n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{-6x\,(3n+1)!}{(3n+4)!} \right| = \lim_{n\to\infty} \left| \frac{-6x}{(3n+4)(3n+3)(3n+2)} \right|$$

$$= 6|x| \lim_{n\to\infty} \left| \frac{1}{(3n+4)(3n+3)(3n+2)} \right| = 0 < 1 \text{ for all } x.$$

$$\boxed{\begin{array}{c}\text{interval of convergence is } (-\infty, \infty) \\ R = \infty\end{array}}$$

2. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges when $x = 5$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following series:



(a) $\sum_{n=0}^{\infty} c_n(-7)^n$

$x-3=-7$

$x=-7+3=-4$

$\boxed{\text{divergent}}$

(b) $\sum_{n=0}^{\infty} c_n 5^n$

$x-3=5 \Rightarrow x=8$

$\boxed{\text{don't know}}$

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$

$x-3=-3 \Rightarrow x=0$

$\boxed{\text{don't know}}$

(d) $\sum_{n=0}^{\infty} c_n 3^n$

$x-3=3 \Rightarrow x=6$

$\boxed{\text{don't know}}$

(e) $\sum_{n=0}^{\infty} c_n(-1)^n$

$\boxed{\text{converges}}$

$x-3=-1$

$x=2$

10.6: Representation of Functions as Power Series

- Geometric Series Formula:

$$\boxed{\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n}, \quad -1 < x < 1. \qquad \boxed{\frac{1}{1-☺} = \sum_{n=0}^{\infty} (☺)^n}, \ |☺|<1$$

- **Term-by-term Differentiation and Integration of power series:**

  If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

  - $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
  - $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

  The radii of convergence of the power series for $f'(x)$ and $\int f(x)\,\mathrm{d}x$ are both R.

3. Find a power series representation for the following functions and determine the interval of convergence.

   (a) $f(x) = \frac{4}{1+x} = 4 \cdot \frac{1}{1-(-x)} = 4 \cdot \sum_{n=0}^{\infty} (-x)^n = 4 \sum_{n=0}^{\infty} (-1)^n x^n$, converges if $|x|<1$

   interval of convergence is $(-1, 1)$

   (b) $f(x) = \frac{4}{2+4x} = \frac{2}{1+2x} = 2 \cdot \frac{1}{1-(-2x)} = 2 \sum_{n=0}^{\infty} (-2x)^n = 2 \sum_{n=0}^{\infty} (-1)^n (2x)^n$

   converges if $|2x|<1$ or $|x|<\frac{1}{2}$

   interval of convergence $-\frac{1}{2} < x < \frac{1}{2}$

(c) $f(x) = \dfrac{-9}{9 - x^4} = -\cancel{9} \cdot \dfrac{1}{\cancel{9}\left(1 - \frac{x^4}{9}\right)} = -\sum_{n=0}^{\infty} \left(\dfrac{x^4}{9}\right)^n = -\sum_{n=0}^{\infty} \dfrac{x^{4n}}{9^n}$

convergent if $\dfrac{x^4}{9} < 1$

or $x^4 < 9$ or $|x| < \sqrt[4]{9}$

interval of convergence: $\underline{-\sqrt{3} < x < \sqrt{3}}$

(d) $f(x) = \ln(3x + 5) \Rightarrow 3x+5>0$ or $x > -\frac{5}{3}$

$\int \dfrac{du}{u} = \ln|u| + C$, $u = 3x+5$, $du = 3dx$

$\int \dfrac{3\,dx}{3x+5} = \ln|3x+5| + C \Rightarrow \ln|3x+5| = C + \int \dfrac{3\,dx}{3x+5}$

$\dfrac{3}{3x+5} = \dfrac{3}{5\left(1+\frac{3x}{5}\right)} = \dfrac{3}{5\left(1-\left(-\frac{3x}{5}\right)\right)} = \dfrac{3}{5}\sum_{n=0}^{\infty}\left(-\dfrac{3x}{5}\right)^n = \dfrac{3}{5}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^n x^n$

$= \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1} x^n$

integrate: $\int \dfrac{3}{3x+5}\,dx = \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\left(\int x^n dx\right)$

$\ln(3x+5) = C + \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\dfrac{x^{n+1}}{n+1}$

plug $x=0$: $\ln 5 = C$

$$\boxed{\ln(3x+5) = \ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\dfrac{x^{n+1}}{n+1}}$$

interval of conv.

$\left|\dfrac{3x}{5}\right| < 1$

$-1 < \dfrac{3x}{5} < 1$

$-\dfrac{5}{3} < x < \dfrac{5}{3}$

$x = \dfrac{5}{3}$: $\sum_{n=0}^{\infty}\dfrac{(-1)^n\left(\frac{3}{5}\right)^{n+1}\left(\frac{5}{3}\right)^{n+1}}{n+1}$

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}$

conv. by A.S.T.

$x = -\dfrac{5}{3}$: $\sum_{n=0}^{\infty}\dfrac{(-1)^n\left(\frac{3}{5}\right)^{n+1}\left(-\frac{5}{3}\right)^{n+1}}{(n+1)}$

$= \sum_{n=0}^{\infty}\dfrac{1}{n+1}$ divergent

by L.C.T.

compare with $\sum_{n=1}^{\infty}\dfrac{1}{n}$

$\left(-\frac{5}{3}, \frac{5}{3}\right]$

(e) $f(x) = x^5 \ln(3x + 5)$

$= x^5\left(\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\dfrac{x^{n+1}}{n+1}\right)$

$= x^5 \ln 5 + x^5\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\dfrac{x^{n+1}}{n+1}$

$$= \boxed{x^5\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{5}\right)^{n+1}\dfrac{x^{n+6}}{n+1}}$$

(f) $f(x) = \dfrac{x^4}{(1-4x)^2}$ $= x^4 \cdot \dfrac{1}{(1-4x)^2}$, $\left(\dfrac{1}{1-4x}\right)' = \dfrac{-4}{-(1-4x)^2} = \dfrac{4}{(1-4x)^2}$

$$\frac{1}{(1-4x)^2} = \frac{1}{4}\left(\frac{1}{1-4x}\right)'$$

$$\frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n = \sum_{n=0}^{\infty} 4^n x^n$$

$$\left(\frac{1}{1-4x}\right)' = \sum_{n=1}^{\infty} 4^n n x^{n-1} = \sum_{k=0}^{\infty} 4^{k+1}(k+1)x^k$$

$n-1=k$

$n=k+1$

$$\frac{x^4}{(1-4x)^2} = \frac{x^4}{4}\cdot\left(\frac{1}{1-4x}\right)' = \frac{x^4}{4}\sum_{n=0}^{\infty} 4^{n+1}(n+1)x^n = \boxed{\sum_{n=0}^{\infty} 4^n (n+1) x^{n+4}}$$

converges if $|4x|<1$ or $|x|<\frac{1}{4}$

$-\frac{1}{4}<x<\frac{1}{4}$

(g) $f(x) = \arctan(16x^4)$

$\int \frac{du}{1+u^2} = \arctan u + C$, $u = 16x^4$, $du = 4(16)x^3\,dx = 64x^3\,dx$

$$\arctan(16x^4) = C + \int \frac{64x^3\,dx}{1+(16x^4)^2}$$

$$\frac{64x^3}{1+256x^8} = 64x^3 \cdot \frac{1}{1-(-256x^8)} = 64x^3 \sum_{n=0}^{\infty} (-256x^8)^n = \overbrace{64}^{2^6}x^3 \sum_{n=0}^{\infty} (-1)^n \overbrace{(256)}^{2^8}{}^n x^{8n}$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^{8n+6} x^{8n+3}$$

$$\arctan(16x^4) = C + \sum_{n=0}^{\infty} (-1)^n 2^{8n+6}\left(\int x^{8n+3}\,dx\right) = C + \sum_{n=0}^{\infty} (-1)^n 2^{8n+6} \frac{x^{8n+4}}{8n+4}$$

plug $x=0$: $\arctan 0 = C \Rightarrow C=0$

$$\boxed{\arctan(16x^4) = \sum_{n=0}^{\infty} (-1)^n 2^{8n+6} \frac{x^{8n+4}}{8n+4}}$$

converges if $|256x^8|<1$ or $x^8 < \frac{1}{256}$ or $|x|<\frac{1}{2}$

$-\frac{1}{2}<x<\frac{1}{2}$

4. Express the integral $\displaystyle\int_{-0.5}^{0} \frac{dx}{1-x^7}$ as a power series.

$$\frac{1}{1-x^7} = \sum_{n=0}^{\infty} (x^7)^n = \sum_{n=0}^{\infty} x^{7n}$$

$$\int_{-0.5}^{0} \frac{dx}{1-x^7} = \sum_{n=0}^{\infty}\left(\int_{-0.5}^{0} x^{7n}\,dx\right) = \sum_{n=0}^{\infty} \frac{x^{7n+1}}{7n+1}\Bigg|_{-0.5}^{0} = -\sum_{n=0}^{\infty} \frac{(-0.5)^{7n+1}}{7n+1}$$

$(-1)^{7n+1}\cdot(0.5)^{7n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{7n+2}(0.5)^{7n+1}}{7n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{7n}(0.5)^{7n+1}}{7n+1}}$$

10.7: Taylor and Maclaurin Series

- **The Taylor series for $f(x)$ about $x = a$:**

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n} =$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- **The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):**

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

- *Known Mclaurin series and their intervals of convergence you must have memorized:*

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + \dots \qquad (1, 1)$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad (-\infty, \infty)$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad (-\infty, \infty)$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad (-\infty, \infty)$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1, 1]$$

Examples.

5. Given that function f has power series expansion (i.e. Taylor series) centered at $a = \pi$. Find this expansion and its radius of convergence if it is given that

$$f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{4^{2n+1}(2n+1)!\,\cancel{n!}}(x-\pi)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^n}{4^{2n+1}(2n+1)!}}$$

do the Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, $\boxed{R = \infty}$

6. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$. $f^{(20)}(0)$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\frac{1}{n!} = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = 1 \text{ for all } n.$$

$$\boxed{f^{(20)}(0) = 1}$$

7. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$f''(x) = 3^2 e^{3x}$

$\cdots$

$f^{(n)}(x) = 3^n e^{3nx}$

$f^{(n)}\left(\frac{1}{3}\right) = 3^n e^{3 \cdot \frac{1}{3}} = 3^n \cdot e$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{1}{3}\right)}{n!} \left(x - \frac{1}{3}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{3^n \cdot e}{n!} \left(x - \frac{1}{3}\right)^n}$$

$\boxed{R = \infty}$

8. Find Taylor series for $f(x) = \dfrac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = (-1)x^{-2}$

$f''(x) = (-1)(-2)x^{-3}$

$f'''(x) = (-1)(-2)(-3)x^{-4}$

$f^{(n)}(x) = (-1)^n \, n! \, x^{-n-1} = \dfrac{(-1)^n \cdot n!}{x^{n+1}}$

$f^{(n)}(5) = \dfrac{(-1)^n n!}{5^{n+1}}$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{5^{n+1} \cdot n!}(x-5)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}(x-5)^n}$$

Do the Ratio Test for $a_n = \dfrac{(-1)^n}{5^{n+1}}(x-5)^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|}{5} < 1 \Rightarrow |x-5| < 5 \Rightarrow \boxed{R = 5}$$

9. Find Maclaurin series for the following functions:

(a) $f(x) = x^3 \sin(x^5)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^5) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^5)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+5}}{(2n+1)!}$$

$$x^3 \sin(x^5) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+5+3}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+8}}{(2n+1)!}}$$

(b) $f(x) = \sin^2 x = \frac{1-\cos 2x}{2}$

(c) $x + 3x^2 + xe^{-x}$

10. Express $\int \frac{\sin(3x)}{x}\, dx$ as an infinite series.

11. Find the sum of the series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{7^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

12. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\, dx$ with error less than 10^{-3}.