Sections 10.5, 10.6, 10.7

10.5: Power Series

- For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:
  1. There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the **radius of convergence.**
  2. The series converges only for $x = a$ (then $R = 0$).
  3. The series converges for all x (then $R = \infty$).
- We find the radius of convergence using the **Ratio Test.**
- An **interval of convergence** is the interval of all x's for which the power series converges.
- You must check the endpoints $x = a \pm R$ individually to determine whether or not they are in the interval of convergence.

1. For the following series find the radius and interval of convergence.

   (a) $\sum_{n=0}^{\infty} \frac{n^4 x^n}{7^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{8^n (x+4)^{3n}}{n^3+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n(5x-3)^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!(x-1)^{n+1}}{4^{n+1}}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-6)^n x^n}{(3n+1)!}$

2. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges when $x = 5$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following series:

(a) $\sum_{n=0}^{\infty} c_n(-7)^n$

(b) $\sum_{n=0}^{\infty} c_n 5^n$

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$

(d) $\sum_{n=0}^{\infty} c_n 3^n$

(e) $\sum_{n=0}^{\infty} c_n(-1)^n$

10.6: Representation of Functions as Power Series

- Geometric Series Formula:

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

- **Term-by-term Differentiation and Integration of power series:**

  If $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

  - $\displaystyle f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
  - $\displaystyle \int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

  The radii of convergence of the power series for $f'(x)$ and $\displaystyle\int f(x)\,\mathrm{d}x$ are both R.

3. Find a power series representation for the following functions and determine the interval of convergence.

   (a) $f(x) = \dfrac{4}{1+x}$

   (b) $f(x) = \dfrac{4}{2+4x}$

(c) $f(x) = \dfrac{-9}{9 - x^4}$

(d) $f(x) = \ln(3x + 5)$

(e) $f(x) = x^5 \ln(3x + 5)$

(f) $f(x) = \dfrac{x^4}{(1-4x)^2}$

(g) $f(x) = \arctan(16x^4)$

4. Express the integral $\displaystyle\int_{-0.5}^{0} \frac{\mathrm{d}x}{1-x^7}$ as a power series.

10.7: Taylor and Maclaurin Series

- **The Taylor series for $f(x)$ about $x = a$:**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
- **The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):**
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
- *Known Mclaurin series and their intervals of convergence you must have memorized:*

$$\begin{array}{ccccccc}
\frac{1}{1-x} & = & \sum_{n=0}^{\infty} x^n & = & 1 + x + x^2 + x^3 + \dots & (1,1) \\
e^x & = & \sum_{n=0}^{\infty} \frac{x^n}{n!} & = & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots & (-\infty,\infty) \\
\cos x & = & \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} & = & 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots & (-\infty,\infty) \\
\sin x & = & \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} & = & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots & (-\infty,\infty) \\
\arctan x & = & \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} & = & x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & [-1,1]
\end{array}$$

Examples.

5. Given that function f has power series expansion (i.e. Taylor series) centered at $a = \pi$. Find this expansion and its radius of convergence if it is given that
$$f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}.$$

6. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$.

7. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?

8. Find Taylor series for $f(x) = \dfrac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?

9. Find Maclaurin series for the following functions:

   (a) $f(x) = x^3 \sin x^5$

   (b) $f(x) = \sin^2 x$

   (c) $x + 3x^2 + xe^{-x}$

10. Express $\displaystyle\int \frac{\sin(3x)}{x}\,\mathrm{d}x$ as an infinite series.

11. Find the sum of the series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{7^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

12. Use series to approximate the integral $\int_0^{0.5} x^2 e^{-x^2}\,\mathrm{d}x$ with error less than 10^{-3}.