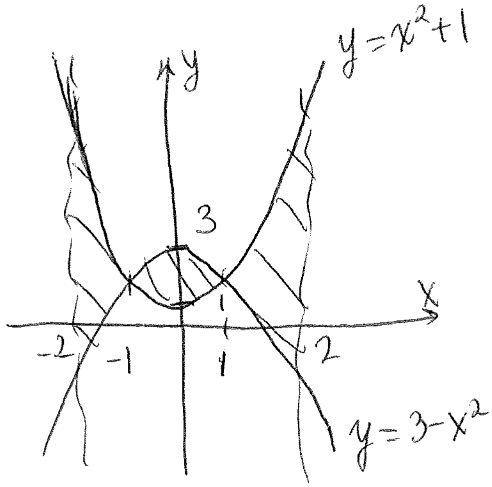
MATH152, 525–530, 534–536 Spring 2013,
Solutions to sample problems for the final

1. Find the area of the region bounded by $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, and $x = 2$.



The region is symmetric with respect to the y-axis.
Points of intersection:

$x^2+1 = 3 - x^2$

$2x^2 = 2$

$x = \pm 1$

$$A(D) = 2\left[\int_0^1 (3-x^2-(x^2+1))\,dx + \int_1^2 (x^2-1-(3-x^2))\,dx\right]$$

$$= 2\left[\int_0^1 (2-2x^2)\,dx + \int_1^2 (2x^2-2)\,dx\right]$$

$$= 2\left[\left(2x - \frac{2x^3}{3}\right)\Big|_0^1 + \left(\frac{2x^3}{3} - 2x\right)\Big|_1^2\right]$$

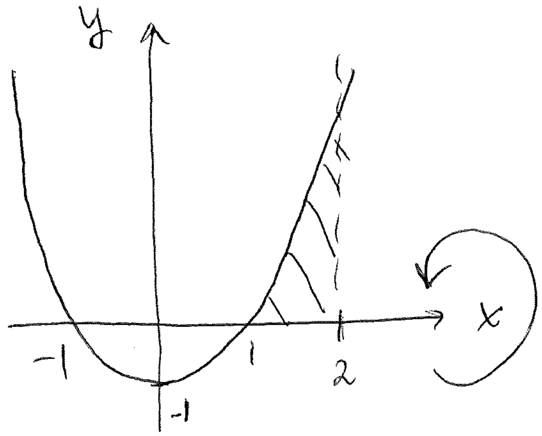
$$= 2\left[2 - \frac{2}{3} + \frac{2\cdot 8}{3} - 4 - \frac{2}{3} + 2\right]$$

$$= 2\left[-\frac{4}{3} + \frac{16}{3}\right]$$

$$= 2(4)$$

$$= \boxed{8}$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 1$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.



disks:

$$V = \pi \int_1^2 [x^2 - 1]^2 \, dx$$

$$= \pi \int_1^2 (x^4 - 2x^2 + 1) \, dx$$

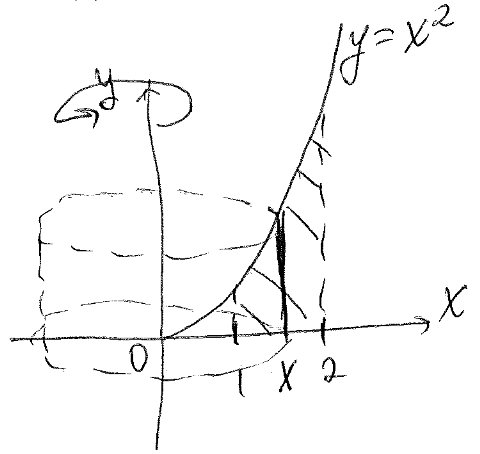
$$= \pi \left(\frac{x^5}{5} - 2\frac{x^3}{3} + x \right) \Big|_1^2$$

$$= \pi \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1 \right)$$

$$= \boxed{\frac{38\pi}{15}}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$, $x = 1$, $x = 2$ about

(a) the y-axis



cylindrical shells:

$1 \le x \le 2$.

$r = x$, $h = x^2$

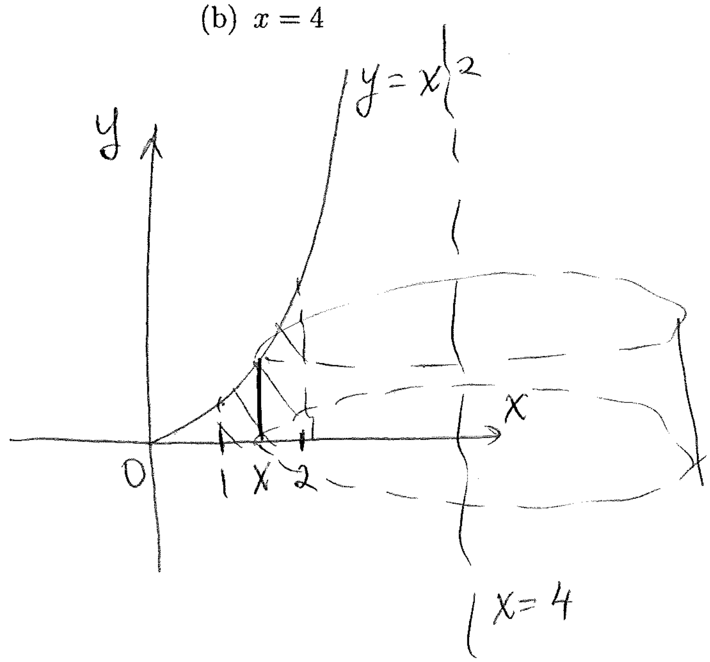
$$V = 2\pi \int_1^2 r h \, dx$$

$$= 2\pi \int_1^2 x x^2 \, dx$$

$$= 2\pi \int_1^2 x^3 \, dx$$

$$= 2\pi \left(\frac{x^4}{4} \right) \Big|_1^2 = \boxed{\frac{15\pi}{2}}$$

(b) $x = 4$



cylindrical shells:

$1 \le x \le 2$

$r = 4 - x$, $h = x^2$

$$V = 2\pi \int_1^2 (4 - x) x^2 \, dx$$

$$= 2\pi \int_1^2 (4x^2 - x^3) \, dx$$

$$= 2\pi \left(\frac{4x^3}{3} - \frac{x^4}{4} \right) \Big|_1^2$$

$$= 2\pi \left(\frac{32}{3} - \frac{16}{4} - \frac{4}{3} + \frac{1}{4} \right)$$

$$= \boxed{\frac{67\pi}{6}}$$

4. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the rope to the top of the building?

a portion of the rope from x to $x+\Delta x$ weighs $0.5\Delta x$ (lb) and must be lifted x (ft). $0 \le x \le 50$

$$W = \int_0^{50} 0.5x\,dx = \frac{x^2}{4}\Big|_0^{50} = \frac{1}{4}(2500) = \boxed{625\,(\text{ft-lb})}$$

5. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

$F = kx$. $\quad 20\text{ cm} \to 0$

$30\text{ cm} \to 30-20 = 10(\text{cm}) = 0.1(\text{m})$.

$25 = k(0.1) \to k = 250 \to F = 250x$

$25\text{ cm} \to 25-20 = 5(\text{cm}) = 0.05(\text{m})$

$$W = \int_0^{0.05} 250x\,dx = \frac{250x^2}{2}\Big|_0^{0.05} = 125(0.0025) = \boxed{0.3125\,(\text{J})}$$

6. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

$$f_{ave} = \frac{1}{\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin^2 x \cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$-\frac{\pi}{2} \to \sin\left(-\frac{\pi}{2}\right) = -1$

$\frac{\pi}{4} \to \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$= \frac{1}{\frac{3\pi}{4}} \int_{-1}^{\frac{\sqrt{2}}{2}} u^2\,du = \frac{4}{3\pi}\,\frac{u^3}{3}\Big|_{-1}^{\frac{\sqrt{2}}{2}} = \boxed{\frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8} + 1\right)}$$

7. Evaluate the integral

(a) $\frac{1}{3}\int 3t^2\cos(1-t^3)dt$ $\quad \left| \begin{array}{l} u = 1 - t^3 \\ du = -3t^2 dt \end{array} \right|$

$$= -\frac{1}{3}\int \cos u \, du = -\frac{1}{3}\sin u + C = \boxed{-\frac{1}{3}\sin(1-t^3)+C}$$

(b) $\int \frac{x^2}{\sqrt{1-x}}dx$ $\quad \left| \begin{array}{l} u = 1-x \\ x = 1-u \\ du = -dx \end{array} \right|$

$$= \int \frac{(1-u)^2(-du)}{\sqrt{u}} = -\int (1-2u+u^2)u^{-1/2}du$$

$$= -\int (u^{-1/2} - 2u^{1/2} + u^{3/2})du$$

$$= -\left(2u^{1/2} - 2\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) + C$$

$$= \boxed{-2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C}$$

(c) $\int_0^1 x^2 e^{-x} dx$

$$\left| \begin{array}{ll} f(x) = x^2 & g'(x) = e^{-x} \\ f'(x) = 2x & g(x) = -e^{-x} \end{array} \right|$$

$$= -x^2 e^{-x} \Big|_0^1 + \int_0^1 2x e^{-x} dx$$

$$= -e^{-1} + 2 \int_0^1 x e^{-x} dx \qquad \left| \begin{array}{ll} f(x) = x & g'(x) = e^{-x} \\ f'(x) = 1 & g(x) = -e^{-x} \end{array} \right|$$

$$= -e^{-1} + 2 \left[-x e^{-x} \Big|_0^1 + \int e^{-x} dx \right]$$

$$= -e^{-1} - 2e^{-1} - 2e^{-x} \Big|_0^1$$

$$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2$$

$$= \boxed{2 - 5e^{-1}}$$

(d) $\int \sin^3 x \cos^4 x \, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \\ \sin^2 x = 1 - \cos^2 x = 1 - u^2 \end{array} \right|$

$= \int \sin x \sin^2 x \cos^4 x \, dx = \int \sin x (1 - \cos^2 x) \cos^4 x \, dx$

$= -\int (1-u^2) u^4 \, du = -\int (u^4 - u^6) \, du = -\frac{u^5}{5} + \frac{u^7}{7} + C$

$= \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$

(e) $\int_0^{\pi/8} \sin^2(2x) \cos^3(2x) \, dx = \int_0^{\pi/8} \sin^2(2x) \cos(2x) \cos^2(2x) \, dx$

$= \int_0^{\pi/8} \sin^2(2x) (1 - \sin^2(2x)) \cos(2x) \, dx = \left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos 2x \, dx \\ 0 \to \sin 0 = 0 \\ \frac{\pi}{8} \to \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{array} \right.$

$= \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} u^2 (1 - u^2) \, du = \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} (u^2 - u^4) \, du$

$= \frac{1}{2} \left(\frac{u^3}{3} - \frac{u^5}{5} \right) \Big|_0^{\frac{\sqrt{2}}{2}}$

$= \boxed{\frac{1}{2} \left(\frac{1}{3} \left(\frac{\sqrt{2}}{2} \right)^3 - \frac{1}{5} \left(\frac{\sqrt{2}}{2} \right)^5 \right)}$

(f) $\int \sin^2 x \cos^4 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{8} \int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\,dx$

$= \frac{1}{8} \int (1+2\cos 2x+\cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x)\,dx$

$= \frac{1}{8} \int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\,dx$

$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x\right) - \frac{1}{8}\int \cos^2 2x \, dx - \frac{1}{8}\int \cos^3 2x \, dx$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2}\,dx - \frac{1}{8}\int \cos^2 2x \cos 2x \, dx$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}\left(x + \frac{1}{4}\sin 4x\right) - \frac{1}{(8)(2)}\int 2(1-\sin^2 2x)\cos 2x \, dx$

$u = \sin 2x$
$du = 2\cos 2x \, dx$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{16}\int (1-u^2)\,du$

$= -\frac{1}{64}\sin 4x + \frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{16}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{16}x + \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x$

$- \frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x + C$

$= \boxed{\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C}$

(g) $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$
$0 \to \tan 0 = 0$
$\frac{\pi}{4} \to \tan \frac{\pi}{4} = 1$

$= \int_0^1 u^4 \, du = \left.\frac{u^5}{5}\right|_0^1 = \boxed{\frac{1}{5}}$

(h) $\int \tan x \sec^3 x \; dx$

$$\left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \end{array} \right|$$

$$= \int u^2 du = \frac{u^3}{3} + C = \boxed{\frac{\sec^3 x}{3} + C}$$

(i) $\int \sin 3x \cos x \; dx = \frac{1}{2} \int [\sin(3x - x) + \sin(3x + x)] \, dx$

$$= \frac{1}{2} [\sin 2x + \sin 4x] \, dx$$

$$= \boxed{\frac{1}{2}\left(-\frac{1}{2} \cos 2x - \frac{1}{4} \cos 4x\right) + C}$$

(j) $\int \frac{x^2}{\sqrt{5-x^2}}dx$

$$\left| \begin{array}{l} x = \sqrt{5}\sin t \\ dx = \sqrt{5}\cos t\,dt \\ \sqrt{5-x^2} = \sqrt{5}\cos t \end{array} \right|$$

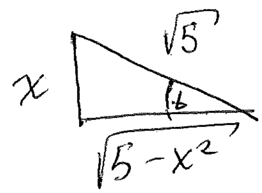
$$= \int \frac{5\sin^2 t\,\sqrt{5}\cos t\,dt}{\sqrt{5}\cos t} = 5\int \sin^2 t\,dt = 5\int \frac{1-\cos 2t}{2}dt$$

$$= \frac{5}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$x = \sqrt{5}\sin t$

$\sin t = \frac{x}{\sqrt{5}}$

$t = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$



$\cos t = \frac{\sqrt{5-x^2}}{\sqrt{5}}$

$$-\frac{1}{2}\sin 2t = -\sin t\cos t = -\frac{x}{\sqrt{5}}\,\frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$= \boxed{\frac{5}{2}\left(\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) - \frac{x\sqrt{5-x^2}}{5}\right) + C}$$

(k) $\int \frac{x^3}{\sqrt{x^2+4}}dx$

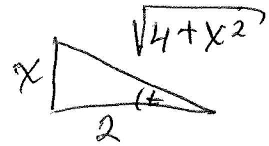
$$\left|\begin{array}{l} x = 2\tan t \\ dx = 2\sec^2 t\,dt \\ \sqrt{x^2+4} = 2\sec t \end{array}\right|$$

$$= \int \frac{8\tan^3 t \; 2\sec^2 t}{2\sec t}dt = 8\int \tan^3 t \sec t \, dt$$

$$\left|\begin{array}{l} u = \sec t \\ du = \sec t \tan t \, dt \\ \tan^2 t = \sec^2 t - 1 \\ \qquad = u^2 - 1 \end{array}\right|$$

$$= 8\int (u^2-1)du = 8\left(\frac{u^3}{3} - u\right) + C$$

$$= 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C$$



$x = 2\tan t$

$\tan t = \frac{x}{2}$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{2}{\sqrt{4+x^2}}} = \frac{\sqrt{4+x^2}}{2}$$

$$= \boxed{8\left(\frac{(4+x^2)^{3/2}}{8} - \frac{\sqrt{4+x^2}}{2}\right) + C}$$

(1) $\int \frac{dx}{\sqrt{x^2+4x-5}}$

$$x^2+4x-5 = (x+2)^2-9$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-9}}$$

$$\left| \begin{array}{l} x+2 = 3\sec t \\ dx = 3\sec t \tan t \, dt \\ \sqrt{(x+2)^2-9} = \sqrt{9\sec^2 t - 9} = 3\tan t \end{array} \right.$$

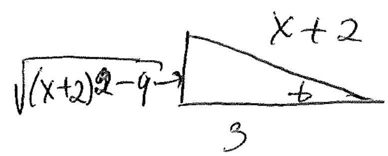
$$= \int \frac{3\sec t \tan t \, dt}{3\tan t \quad dt} = \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C$$

$$x+2 = 3\sec t$$

$$\sec t = \frac{x+2}{3}$$

$$\cos t = \frac{1}{\sec t} = \frac{3}{x+2}$$



$$\tan t = \frac{\sqrt{(x+2)^2-9}}{3}$$

$$= \boxed{\ln\left| \frac{x+2}{3} + \frac{\sqrt{(x+2)^2-9}}{3} \right| + C}$$

(m) $\displaystyle\int \frac{dx}{x^2(x^2+1)}$

Partial fractions:

$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$$

$$= \frac{Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2}{x^2(x^2+1)}$$

$$\frac{1}{x^2(x^2+1)} = \frac{x^3(A+C) + x^2(B+D) + Ax + B}{x^2(x^2+1)}$$

$$1 = x^3(A+C) + x^2(B+D) + Ax + B$$

x^3: $\quad A + C = 0$

x^2: $\quad B + D = 0$

x: $\quad A = 0 \longrightarrow C = -A = 0$

1: $\quad B = 1 \longrightarrow D = -B = -1$

$$\frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$$

$$\int \frac{dx}{x^2(x^2+1)} = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1}$$

$$= \boxed{-\frac{1}{x} - \arctan x + C}$$

(n) $\int \frac{x^2+3x-1}{x-1}dx$

Improper fraction:

$$\begin{array}{r} x+4 \\ x-1 \overline{)\, x^2+3x-1} \\ -\;x^2-x \\ \hline 4x-1 \\ 4x-4 \\ \hline 3 \end{array}$$

$$\frac{x^2+3x-1}{x-1} = x+4+\frac{3}{x-1}$$

$$\int \frac{x^2+3x-1}{x-1}dx = \int \left(x+4+\frac{3}{x-1}\right)dx$$

$$= \boxed{\frac{x^2}{2}+4x+3\ln|x-1|+C}$$

(o) $\int\limits_0^\infty \frac{dx}{(x+2)(x+3)}$

Partial fractions:

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$$1 = A(x+3)+B(x+2)$$

$x=-3$: $\quad 1=-B \longrightarrow B=-1$

$x=-2$: $\quad 1=A$

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int_0^\infty \frac{dx}{(x+2)(x+3)} = \lim_{t\to\infty} \int_0^t \frac{dx}{(x+2)(x+3)}$$

$$= \lim_{t\to\infty} \left[\int_0^t \frac{dx}{x+2} - \int_0^t \frac{dx}{x+3} \right]$$

$$= \lim_{t\to\infty} \left[\ln|x+2| \Big|_0^t - \ln|x+3| \Big|_0^t \right]$$

$$= \lim_{t\to\infty} \left[\ln|t+2| - \ln 2 - \ln|t+3| + \ln 3 \right]$$

$$= \lim_{t\to\infty} \ln\left|\frac{t+2}{t+3}\right| - \ln 2 + \ln 3$$

$$= \ln\Big(\underbrace{\lim_{t\to\infty} \frac{t+2}{t+3}}_{1}\Big) + \ln\frac{3}{2} = \ln 1 + \ln\frac{3}{2} = \boxed{\ln\frac{3}{2}}$$

(p) $\displaystyle\int_{-\infty}^{1} \frac{dx}{(2x-3)^2}$

$$\int_{-\infty}^{1} \frac{dx}{(2x-3)^2} = \lim_{s\to-\infty} \int_{s}^{1} \frac{dx}{(2x-3)^2}$$

$$= \lim_{s\to-\infty} \frac{-1}{2} \frac{1}{(2x-3)} \Big|_{s}^{1}$$

$$= \lim_{s\to-\infty} \left(-\frac{1}{2}\left[\frac{1}{-1} - \frac{1}{2s-3}\right]\right)$$

$$= \frac{1}{2} + \frac{1}{2}\lim_{s\to-\infty} \overset{0}{\frac{1}{2s-3}}$$

$$= \boxed{\frac{1}{2}}$$

(q) $\displaystyle\int_4^5 \frac{dx}{(5-x)^{2/5}} = \lim_{t\to 5^-} \int_4^t \frac{dx}{(5-x)^{2/5}}$

$$= -\lim_{t\to 5^-} \left. \frac{(5-x)^{-2/5+1}}{-\frac{2}{5}+1} \right|_4^t$$

$$= -\lim_{t\to 5^-} \cancel{\frac{5(5-t)^{3/5}}{3}}^{\,0} + \frac{5}{3}$$

$$= \boxed{\frac{5}{3}}$$

8. Find the length of the curve $x(t) = 3t - t^3$, $y(t) = 3t^2$, $0 \le t \le 2$.

$$x'(t) = 3 - 3t^2, \quad y'(t) = 6t$$

$$L = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^2 \sqrt{(3-3t^2)^2 + 36t^2}\, dt$$

$$= \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt$$

$$= \int_0^2 \sqrt{9 + 18t^2 + 9t^4}\, dt$$

$$= 3\int_0^2 \sqrt{1 + 2t^2 + t^4}\, dt$$

$$= 3\int_0^2 \sqrt{(1+t^2)^2}\, dt$$

$$= 3\int_0^2 (1+t^2)\, dt$$

$$= 3\left(t + \frac{t^3}{3}\right)\Big|_0^2$$

$$= 3\left(2 + \frac{8}{3}\right)$$

$$= \boxed{14}$$