8. Evaluate the integral

(a) $\int t^2 \cos(1-t^3)dt$ $\quad \left| \begin{array}{l} u = 1-t^3 \\ du = -3t^2 dt \\ t^2 dt = -\frac{du}{3} \end{array} \right| \quad = -\frac{1}{3}\int \cos u \, du = \ldots$

(b) $\int \frac{x^2}{\sqrt{1-x}}dx$ $\quad \left| \begin{array}{l} x = 1-u \\ u = 1-x \\ du = -dx \end{array} \right| \quad = -\int \frac{(1-u)^2}{\sqrt{u}} du = -\int (1-2u+u^2) u^{-1/2} du = \ldots$

(d) $\int \overbrace{x^3}^{x \cdot x^2} e^{x^2}\, dx$

$$\left| \begin{array}{l} w = x^2 \\ dw = 2x\,dx \\ x\,dx = \frac{dw}{2} \end{array} \right| = \int w e^{w} \frac{dw}{2} = \frac{1}{2} \int w e^{w} dw \left| \begin{array}{ll} u = w & v' = e^{w} \\ u' = 1 & v = e^{w} \end{array} \right|$$

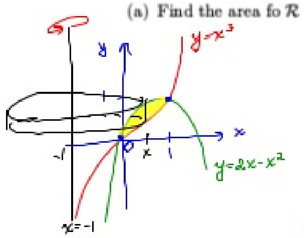
$$= \frac{1}{2}\left(w e^{w} - \int e^{w} dw\right) = \frac{1}{2}\left(w e^{w} - e^{w}\right) + C$$

$$= \boxed{\frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C}$$

Review for Test 1.

1. Let $\mathcal{R}$ be the region in the <u>first quadrant</u> bounded by the curves $y = x^3$ and $y = 2x - x^2$.

   (a) Find the area fo $\mathcal{R}$



Points of intersection:

$x^3 = 2x - x^2$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0$

$x = 0,\ x = -2,\ x = 1$

$$A = \int_0^1 \left([\text{top}] - [\text{bottom}]\right)dx = \int_0^1 (2x - x^2 - x^3)\,dx$$

$$= \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \boxed{\frac{5}{12}}$$

   (b) Find the volume obtained by rotating $\mathcal{R}$ about the [redacted] line $x = -1$.

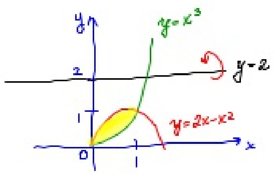
Shells.

$[\text{radius}] = x + 1$

$[\text{height}] = 2x - x^2 - x^3$

$$V = 2\pi \int_0^1 (x+1)(2x - x^2 - x^3)\,dx = \ldots$$

   (c) Find the volume obtained by rotating $\mathcal{R}$ about the [redacted] line $y = 2$.



Washers.

$[\text{inner radius}] = 2 - (2x - x^2)$

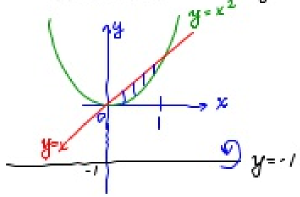
$[\text{outer radius}] = 2 - x^3$

$$V = \pi \int_0^1 \left([\text{outer radius}]^2 - [\text{inner radius}]^2\right)dx$$

$$= \pi \int_0^1 \left[(2 - x^3)^2 - (2 - 2x + x^2)^2\right] dx = \ldots$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about

(a) the [redacted] line $y=-1$.



washers.

$[\text{inner radius}] = x^2+1$

$[\text{outer radius}] = x+1$

$V = \pi \int_0^1 \left((x+1)^2 - (x^2+1)^2 \right) dx = \ldots$

(b) the y-axis

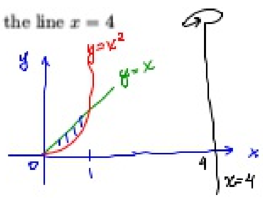
shells.

$[\text{radius}] = x$

$[\text{height}] = [\text{top}] - [\text{bottom}] = x - x^2$

$V = 2\pi \int_0^1 x(x-x^2)\, dx = \ldots$

(c) the line $x = 4$

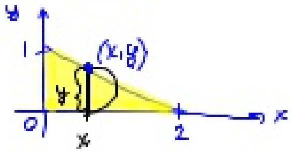


shells: $[\text{radius}] = 4-x$

$[\text{height}] = x - x^2$

$V = 2\pi \int_0^1 (4-x)(x-x^3)\, dx = \ldots$

3. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.



similar triangles:

$$\frac{y}{1} = \frac{2-x}{2}$$

$$y = 1 - \frac{x}{2}$$

Equation of the line through (2,0) and (0,1)

slope $= -\frac{1}{2}$

$$y - 0 = -\frac{1}{2}(x-2)$$

$$y = -\frac{x}{2} + 1$$

Integrate for x, $0 \le x \le 2$.

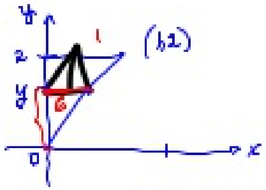
$$A = \frac{1}{2}\pi \cdot \left(\frac{y}{2}\right)^2 = \frac{\pi}{8} y^2$$

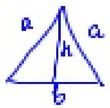
Express y in terms of x.

$$A(x) = \frac{\pi}{8}\left(1 - \frac{x}{2}\right)^2$$

$$V = \int_0^2 A(x)\,dx = \frac{\pi}{8}\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \ldots$$

6. Find the volume of the solid whose base is the triangular region with endpoints $(0,0)$, $(0,2)$, and $(1,2)$, and where cross sections perpendicular to the y-axis are isosceles triangles with height equal to base.





integrate for $0 \le y \le 2$.

$A = \frac{1}{2} b h = \frac{1}{2} b^2$

Express b in terms of y

$\frac{b}{1} = \frac{y}{2} \Rightarrow b = \frac{y}{2}$

$A(y) = \frac{1}{2}\left(\frac{y}{2}\right)^2 = \frac{1}{8} y^2$

$V = \int_0^2 \frac{1}{8} y^2 \, dy = \ldots$

5. A spring has a natural length of 20 cm. If a 10 J work is required to keep it stretched to a length 25 cm, how much work is done in stretching the spring from 30 cm to 80 cm?

20 cm is the natural length.

25 cm is 5 cm beyond the natural length.

5 cm = 0.05 m

$f(x) = kx$, k is an unknown const.

$$10 = \int_0^{0.05} kx\,dx$$

$$10 = \frac{kx^2}{2}\Big|_0^{0.05}$$

$$10 = \frac{0.0025\,k}{2} \Rightarrow k = \frac{20}{0.0025} = \frac{100 \cdot 2000}{25} = 8000$$

$f(x) = 8000x$

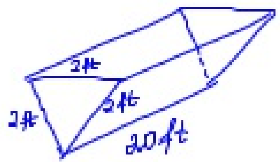
$30 \to 30 - 20 = 10$ cm $= 0.1$ m

$80 \to 80 - 20 = 60$ cm $= 0.6$ m

$$W = \int_{0.1}^{0.6} 8000x\,dx = 4000x^2\Big|_{0.1}^{0.6} = 4000(0.36 - 0.01)$$

$$= 4000(0.35) = \boxed{1400\ (J)}$$

6. A tank of water is 20 ft long and has a vertical cross section in a shape of an equilateral triangle with sides 2 ft long. The tank is filled with water to a depth of 18 inches. Determine the amount of work needed to pump all of the water to the top of the tank. The weight of water is 62.5 lb/ft^3 .



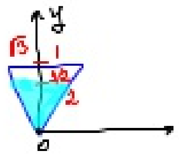
$0 \le y \le \frac{18}{12} = \frac{3}{2}$ ft

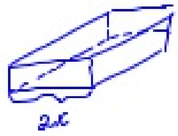
height of the tank $= \sqrt{2^2-1^2} = \sqrt{3}$

Take a slice of water between y and $y+\Delta y$.

it is a "box" of height Δy, length 20

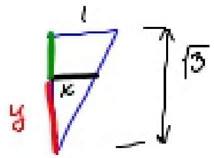
width $2x$.





Volume of the slice $= (2x)(20)(\Delta y)$

Express x in terms of y.



$\frac{x}{1} = \frac{y}{\sqrt{3}}$

Volume $= \frac{2y}{\sqrt{3}}(20)\Delta y$

weight of the slice $= (\text{volume})(62.5)$

$= \frac{40y}{\sqrt{3}}(62.5)\Delta y$

distance traveled by the slice is $\sqrt{3}-y$

work done $= \int_0^{3/2} [\text{weight}][\text{distance}]\, dy$

$= \int_0^{3/2} \frac{40y}{\sqrt{3}}(62.5)(\sqrt{3}-y)\, dy = \frac{40(62.5)}{\sqrt{3}} \int_0^{3/2} y(\sqrt{3}-y)\, dy$

7. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x\,dx = \frac{4}{3\pi} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$x = -\frac{\pi}{2} \Rightarrow u = -1$

$x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}$

$$= \frac{4}{3\pi} \int_{-1}^{\sqrt{2}/2} u^2\,du = \frac{4}{3\pi} \left.\frac{u^3}{3}\right|_{-1}^{\sqrt{2}/2}$$

$$= \frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8} - (-1)\right) = \boxed{\frac{4}{9\pi}\left(\frac{\sqrt{2}}{4} + 1\right)}$$