Examples. Find the integral

1. $\int \sin^3 x \cos^4 x\, dx$ $= \int \sin x \overbrace{\sin^2 x}^{1-\cos^2 x} \cos^4 x\, dx = \int \sin x (1-\cos^2 x) \cos^4 x\, dx$ $\quad \begin{array}{|l} u = \cos x \\ du = -\sin x\, dx \end{array}$

$= -\int (1-u^2) u^4\, du = -\int (u^4 - u^6)\, du = -\frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$

2. $\int_0^{\pi/8} \sin^2(2x) \cos^5(2x)\, dx$ $= \int_0^{\pi/8} \sin^2(2x) \cos(2x) \overbrace{\cos^4(2x)}^{[\cos^2(2x)]^2} dx = \int_0^{\pi/8} \sin^2(2x) \cos(2x) [1-\sin^2(2x)]^2\, dx$ $\quad \begin{array}{|l} u = \sin(2x) \\ du = 2\cos 2x\, dx \\ \cos 2x\, dx = \frac{du}{2} \\ x = 0 \Rightarrow u = \sin 0 = 0 \\ x = \frac{\pi}{8} \Rightarrow u = \sin\frac{2\pi}{8} = \sin\frac{\pi}{4} = \sqrt{2}/2 \end{array}$

$= \int_0^{\sqrt{2}/2} u^2 (1-u^2)^2 \frac{du}{2} = \frac{1}{2} \int_0^{\sqrt{2}/2} u^2 (1 - 2u^2 + u^4)\, du = \frac{1}{2} \int_0^{\sqrt{2}/2} (u^2 - 2u^4 + u^6)\, du$

$= \frac{1}{2} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right)\Big|_0^{\sqrt{2}/2} = \frac{1}{2} \left(\frac{1}{3} \underbrace{\left(\frac{\sqrt{2}}{2}\right)^3}_{\frac{2\sqrt{2}}{8}} - \frac{2}{5} \underbrace{\left(\frac{\sqrt{2}}{2}\right)^5}_{\frac{4\sqrt{2}}{32}} + \frac{1}{7} \underbrace{\left(\frac{\sqrt{2}}{2}\right)^7}_{\frac{8\sqrt{2}}{128}} \right) = \boxed{\frac{1}{2}\left(\frac{1}{3} \frac{\sqrt{2}}{4} - \frac{2}{5} \frac{\sqrt{2}}{8} + \frac{1}{7} \frac{\sqrt{2}}{16} \right)}$

$$\boxed{\int \cos ax\,dx = \frac{1}{a}\sin ax + C \quad \int \sin bx\,dx = -\frac{1}{a}\cos bx + C}$$

3. $\int \sin^2 x \cos^4 x\, dx = \int (\sin^2 x \cos^2 x)\cos^2 x\, dx = \int \underbrace{(\sin x \cos x)^2}_{\frac{1}{2}\sin 2x} \underbrace{\cos^2 x}_{\frac{1+\cos 2x}{2}} dx = \int \frac{1}{4}\sin^2(2x)\,\frac{1+\cos 2x}{2}\,dx$

$= \frac{1}{8}\int \sin^2(2x)(1+\cos 2x)\,dx = \frac{1}{8}\int \sin^2 2x\,dx + \frac{1}{8}\int \sin^2 2x \cos 2x\,dx$

$u = \sin 2x$
$du = 2\cos 2x\,dx$
$\cos 2x\,dx = \frac{du}{2}$

$= \frac{1}{8}\int \frac{1-\cos 4x}{2}\,dx + \frac{1}{8}\int u^2 \frac{du}{2}$

$= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x\right) + \frac{1}{16}\,\frac{u^3}{3} + C = \boxed{\frac{1}{16}\left(x - \frac{1}{4}\sin 4x\right) + \frac{1}{48}\sin^3 2x + C}$

4. $\int_0^{\pi/4} \tan^4 x \sec^4 x\, dx = \int_0^{\pi/4} \tan^4 x \underbrace{\sec^2 x}_{\tan^2 x + 1} \sec^2 x\, dx = \int_0^{\pi/4} \tan^4 x(\tan^2 x + 1)\sec^2 x\, dx$

$u = \tan x$
$du = \sec^2 x\, dx$
$x = 0 \Rightarrow u = \tan 0 = 0$
$x = \pi/4 \Rightarrow u = \tan\frac{\pi}{4} = 1$

$= \int_0^1 u^4(u^2+1)\,du = \int_0^1 (u^6 + u^4)\,du = \left(\frac{u^7}{7} + \frac{u^5}{5}\right)\Big|_0^1 = \frac{1}{7} + \frac{1}{5} = \boxed{\frac{12}{35}}$

5. $\int \tan^3 x \sec^3 x\, dx = \int (\tan x \sec x)\sec^2 x \underbrace{\tan^2 x}_{\sec^2 x - 1}\, dx = \int (\tan x \sec x)\sec^2 x(\sec^2 x - 1)\,dx$

$u = \sec x$
$du = \sec x \tan x\, dx$

$= \int u^2(u^2-1)\,du = \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$

1. Write out the form of the partial fraction decomposition (do not try to solve)

$$\frac{20x^3+12x^2+x}{(x^3-x)(x^3+2x^2-3x)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x(x^2-1)x(x^2+2x-3)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)(x+1)(x+3)(x-1)(x^2+x+1)(x^2+9)^2}$$

$$= \frac{20x^3+12x^2+x}{x^2(x-1)^2(x+1)(x+3)(x^2+x+1)(x^2+9)^2} = \boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}+\frac{D}{(x-1)^2}+\frac{E}{x+1}+\frac{F}{x+3}+\frac{Gx+H}{x^2+x+1}+\frac{Ix+J}{x^2+9}+\frac{Kx+L}{(x^2+9)^2}}$$

2. Evaluate the integral

(a) $\int (4x^2-25)^{-3/2}\,dx$

$\sqrt{a^2-x^2}$	$x = a\sin t$
$\sqrt{x^2+a^2}$	$x = a\tan t$
$\sqrt{x^2-a^2}$	$x = a\sec t$

$2x = 5\sec t$ or $x = \frac{5}{2}\sec t$

$dx = \frac{5}{2}\sec t\tan t\,dt$

$(4x^2-25)^{1/2} = \sqrt{4x^2-25} = \sqrt{25\sec^2 t-25}$

$= \sqrt{25(\sec^2 t-1)} = \sqrt{25\tan^2 t} = 5\tan t$

$$= \int \underbrace{\frac{1}{(5\tan t)^3}}_{\frac{1}{(\sqrt{4x^2-25})^3}} \cdot \underbrace{\frac{5}{2}\sec t\tan t\,dt}_{dx}$$

$$= \frac{5}{2}\cdot\frac{1}{125}\int\frac{\sec t\tan t\,dt}{\tan^3 t}$$

$$= \frac{1}{50}\int\frac{\sec t\,dt}{\tan^2 t} = \frac{1}{50}\int\frac{\frac{1}{\cos t}\,dt}{\frac{\sin^2 t}{\cos^2 t}}$$

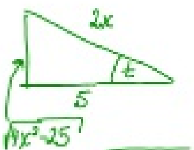
$$= \frac{1}{50}\int\frac{1}{\cancel{\cos t}}\cdot\frac{\cos^{\cancel{2}} t}{\sin^2 t}\,dt = \frac{1}{50}\int\frac{\cos t}{\sin^2 t}\,dt \qquad \begin{vmatrix} u = \sin t \\ du = \cos t\,dt \end{vmatrix}$$

$$= \frac{1}{50}\int\frac{du}{u^2} = \frac{1}{50}\left(-\frac{1}{u}\right)+C = -\frac{1}{50\sin t}+C$$

$$\boxed{= -\frac{1}{50}\cdot\frac{2x}{\sqrt{4x^2-25}}+C}$$

$\sec t = \frac{2x}{5}$

$\cos t = \frac{5}{2x}$



$\sin t = \frac{\sqrt{4x^2-25}}{2x}$

(b) $\int \frac{(x-1)^2}{5\sqrt{24-x^2+2x}}dx$

complete the square: $24-x^2+2x = 24-(x^2-2x) = 24-(x^2-2x+1-1) = 24-(x^2-2x+1)-(-1)$

$= 25-(x-1)^2$

$= \int \frac{(x-1)^2}{5\sqrt{25-(x-1)^2}}\,dx$

$x-1 = 5\sin t$ or $x = 1+5\sin t$

$dx = 5\cos t\,dt$

$\sqrt{25-(x-1)^2} = \sqrt{25-25\sin^2 t}$

$= \sqrt{25(1-\sin^2 t)} = \sqrt{25\cos^2 t}$

$= 5\cos t$

$\cos t = \frac{\sqrt{25-(x-1)^2}}{5}$

$\sin t = \frac{x-1}{5}$

$t = \arcsin\left(\frac{x-1}{5}\right)$

$= \int \frac{\cancel{5}\,\cancel{25}\sin^2 t}{\cancel{5}\,\cancel{(5\cos t)}}\,\cancel{5\cos t}\,dt$

$= 5\int \sin^2 t\,dt = 5\int \frac{1-\cos 2t}{2}\,dt$

$\sin 2t = 2\sin t\cos t$

$= \frac{5}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$

$= \frac{5}{2}\left(t - \sin t\cos t\right) + C$

$= \boxed{\frac{5}{2}\left(\arcsin\left(\frac{x-1}{5}\right) - \frac{(x-1)\sqrt{25-(x-1)^2}}{25}\right) + C}$

(6*) $\int \frac{x^3\,dx}{\sqrt{x^2+1}} = \int \frac{x^2\cdot x\,dx}{\sqrt{x^2+1}}$

$u = x^2+1,\ x^2 = u-1$

$du = 2x\,dx$

$x\,dx = \frac{du}{2}$

$\int \frac{(u-1)\,\frac{du}{2}}{\sqrt{u}}$

$x = \tan t$

(c) $\int \frac{5x^2+x+12}{x^3+4x}dx$

$$\frac{5x^2+x+12}{x^3+4x} = \frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{5x^2+x+12}{\cancel{x^3+4x}} = \frac{A(x^2+4)+x(Bx+C)}{\cancel{x(x^2+4)}}$$

$$5x^2+x+12 = Ax^2+4A+Bx^2+Cx$$

$$5x^2+x+12 = (A+B)x^2+Cx+4A$$

$$\begin{array}{ll} x^2: & 5 = A+B \\ x: & 1 = C \\ 1: & 12 = 4A \end{array} \quad B = 5-A = \boxed{2=B} \quad \boxed{C=1} \quad \Rightarrow \boxed{A=3}$$

$$\int \frac{5x^2+x+12}{x^3+4x}dx = \frac{3}{x} + \frac{2x+1}{x^2+4} = \int\left[\frac{3}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\right]dx$$

$u = x^2+4$

$du = 2x\,dx$

$$= 3\ln|x| + \int\frac{du}{u} + \frac{1}{2}\arctan\frac{x}{2}+C$$

$$= 3\ln|x| + \ln|u| + \frac{1}{2}\arctan\frac{x}{2}+C$$

$$= \boxed{3\ln|x| + \ln|x^2+4| + \frac{1}{2}\arctan\frac{x}{2}+C}$$

$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ | $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$ if $f(x)$ is discont. @ a | $\int_1^\infty \frac{dx}{x^p}$ converges, if $p>1$ | $\int_0^1 \frac{dx}{x^p}$ converges, if $p<1$

3. Determine whether the given integral is convergent or divergent.

Comparison Thm. If $f(x) \geq g(x)$ on (a,∞) then

(a) if $\int_a^\infty f(x)dx$ is conv, then $\int_a^\infty g(x)dx$ is convergent as well.

(b) if $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent as well.

(a) $\displaystyle\int_1^\infty \frac{4+\cos^4 x}{x}\,dx$

$0 \leq \cos^4 x \leq 1$

$\frac{4}{x} \leq \frac{4+\cos^4 x}{x} \leq \frac{5}{x}$

By Comparison Thm. $\int_1^\infty \frac{4+\cos^4 x}{x}dx$ diverge by comparison with $\int_1^\infty \frac{4}{x}dx$ $(p=1)$

(b) $\displaystyle\int_1^\infty \frac{3+\sin x}{x^2}\,dx$

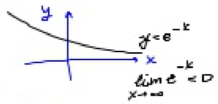
$-1 \leq \sin x \leq 1$

$-1+3 \leq 3+\sin x \leq 3+1$

$\frac{2}{x^2} \leq \frac{3+\sin x}{x^2} \leq \frac{4}{x^2}$

$\int_1^\infty \frac{3+\sin x}{x^2}dx$ conves by comparison with $\int_1^\infty \frac{4}{x^2}dx$ $(p=2>1)$

(c) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}}\,dx$



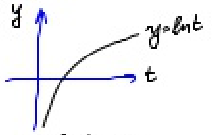
$\frac{1}{\sqrt{x}+e^{4x}} \leq \frac{1}{e^{4x}}$

$\int_0^\infty e^{-4x}dx = \lim_{t\to\infty}\int_0^t e^{-4x}dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_0^t = -\frac{1}{4}\lim_{t\to\infty}\left(\overset{0}{e^{-4t}} - e^0\right) = \frac{1}{4}$ convergent.

$\int_0^\infty \frac{dx}{\sqrt{x}+e^{4x}}$ converges by comparison with $\int_0^\infty e^{-4x}dx$

4. Compute the following integrals or show that they diverge.

(a) $\displaystyle\int_e^\infty \frac{dx}{x\,\ln^5 x} = \lim_{t\to\infty} \int_e^t \frac{dx}{x\,\ln^5 x}$ $\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ x = e \Rightarrow u = \ln e = 1 \\ x = t \Rightarrow u = \ln t \end{array} \right|$ $= \displaystyle\lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^5}$ $p = 5 > 1$ converges.

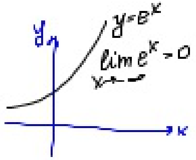


$\displaystyle\lim_{t\to\infty} \ln t = \infty$

$$= \lim_{t\to\infty} \frac{u^{-5+1}}{-5+1}\Big|_1^{\ln t} = \lim_{t\to\infty} \frac{u^{-4}}{-4}\Big|_1^{\ln t} = \lim_{t\to\infty}\left(-\cancel{\frac{1}{4\ln^4 t}}^{\,0} + \frac{1}{4}\right) = \boxed{\frac{1}{4}}$$

(b) $\displaystyle\int_{-\infty}^{0} (1+x)e^x\,dx = \lim_{t\to-\infty} \int_t^0 (1+x)e^x\,dx$ Integrate by parts: $\left| \begin{array}{ll} u = 1+x & v' = e^x \\ u' = 1 & v = e^x \end{array} \right|$

$$= \lim_{t\to-\infty}\left[(1+x)e^x\Big|_t^0 - \int_t^0 e^x\,dx\right] = \lim_{t\to-\infty}\left[1 - (1+t)e^t - e^x\Big|_t^0\right]$$



$$= \lim_{t\to-\infty}\left[\cancel{1} - (\cancel{1}+t)e^t - \cancel{1} + \cancel{e^t}\right]$$

$$= -\lim_{t\to-\infty} te^t \;|0\cdot\infty| = -\lim_{t\to-\infty}\frac{t}{e^{-t}} \;\underset{\text{d'Hospital's Rule}}{=}\; -\lim_{t\to-\infty}\frac{1}{-e^{-t}}$$

$$= +\lim_{t\to-\infty} e^t = \boxed{0} \quad \boxed{\text{converges}}$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{6x^5}{(x^6+3)^3}\,dx = \int_{-\infty}^{0} + \int_{0}^{\infty} = \lim_{t\to-\infty} \int_t^0 \frac{5x^4}{(x^5+3)^3}\,dx + \lim_{s\to\infty} \int_0^s \frac{5x^4}{(x^5+3)^3}\,dx$

$$\left|\begin{array}{l} u = x^6+3 \\ du = 6x^5\,dx \\ x=t \Rightarrow u = t^6+3 \\ x=s \Rightarrow u = s^6+3 \\ x=0 \Rightarrow u = 3 \end{array}\right|$$

$(p = 3 > 1)$ convergent.

$$= \lim_{t\to-\infty} \int_{t^6+3}^{3} \frac{du}{u^3} + \lim_{s\to\infty} \int_{3}^{s^6+3} \frac{du}{u^3}$$

$$= \lim_{t\to-\infty} \left.\frac{u^{-2}}{-2}\right|_{t^6+3}^{3} + \lim_{s\to\infty} \left.\frac{u^{-2}}{-2}\right|_{3}^{s^6+3}$$

$$= -\frac{1}{2(3^2)} + \frac{1}{2}\lim_{t\to-\infty} \cancel{\frac{1}{(t^6+3)^2}}^{\,0} - \frac{1}{2}\lim_{s\to\infty} \cancel{\frac{1}{(s^6+3)^2}}^{\,0} + \frac{1}{2(3^2)} = 0$$

(d) $\displaystyle\int_0^{2019} \frac{1}{\sqrt{2019-x}}\,dx$ discontinuity @ $x=2019$

$p = \frac{1}{2} < 1$ convergent

$$= \lim_{t\to 2019^-} \int_0^t \frac{1}{\sqrt{2019-x}}\,dx$$

$$\left|\begin{array}{l} u = 2019-x \\ du = -\,dx \\ x=0 \Rightarrow u = 2019 \\ x=t \Rightarrow u = 2019-t \end{array}\right|$$

$$= -\lim_{t\to 2017^-} \int_{2019}^{2019-t} u^{-1/2}\,du$$

$$= -\lim_{t\to 2019^-} \left.\frac{u^{1/2}}{1/2}\right|_{2019}^{2019-t} = -2\left(\lim_{t\to 2019^-} \cancel{\sqrt{2019-t}}^{\,0} - \sqrt{2019}\right) = \boxed{2\sqrt{2019}}$$