MATH 251, Section ______
Thursday, Oct. 28, 2010

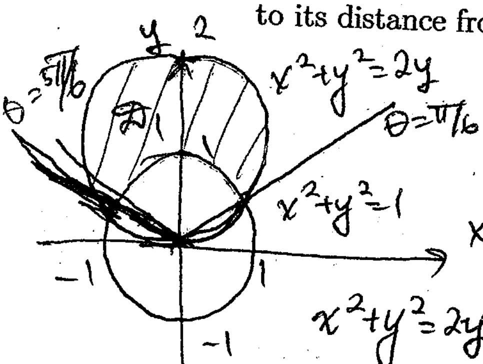
Quiz 9 (Sections 13.6, 13.8).
Dr. M. Vorobets

NAME (print): key

Due Tuesday, Nov. 2 at the beginning of class.

No credit for unsupported answers will be given. Clearly indicate your final answer.

1. *[5 pts.]* A lamina occupies the region inside the circle $x^2+y^2=2y$ but outside the circle $x^2+y^2=1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.



~~Polar~~ $\rho(x,y)=\dfrac{k}{\sqrt{x^2+y^2}}$, k is a constant.

~~Polar~~

$m=\iint_D \rho(x,y)\,dA=\left|\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \\ 1\le r\le 2\sin\theta \\ \pi/6\le\theta\le 5\pi/6 \\ dA=r\,dr\,d\theta\end{array}\right| =\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta}\frac{k}{r}\,r\,dr\,d\theta$

$=k\int_{\pi/6}^{5\pi/6}(2\sin\theta-1)\,d\theta$

$=k(-2\cos\theta-\theta)\Big|_{\pi/6}^{5\pi/6}$

$=k\left(-2\left(-\frac{\sqrt3}{2}-\frac{\sqrt3}{2}\right)-\frac{5\pi}{6}+\frac{\pi}{6}\right)$

$\boxed{2k(\sqrt3-\pi/3)}$

$x^2+y^2=2y\to r=2\sin\theta$

$x^2+y^2=1\to r=1$

intersection $2\sin\theta=1$

$\theta_1=\pi/6,\ \theta_2=5\pi/6.$

$\rho(x,y)=\dfrac{k}{\sqrt{x^2+y^2}}=\dfrac{k}{r}$

By symmetry of D $\bar x=0$.

~~polar~~ coordinates

$\bar y=\frac1m\iint_D y\rho(x,y)\,dA=\frac{1}{2k(\sqrt3-\pi/3)}\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} r\sin\theta\,\frac{k}{r}\,r\,dr\,d\theta=$

$=\frac{1}{2k(\sqrt3-\pi/3)}\int_{\pi/6}^{5\pi/6}\sin\theta\,k\frac{r^2}{2}\Big|_1^{2\sin\theta}d\theta=\frac{1}{2k(\sqrt3-\pi/3)}\frac{k}{2}\int_{\pi/6}^{5\pi/6}\sin\theta(4\sin^2\theta-1)\,d\theta$

$=\frac{1}{4(\sqrt3-\pi/3)}\int_{\pi/6}^{5\pi/6}(4\sin^3\theta-\sin\theta)\,d\theta=\frac{1}{4(\sqrt3-\pi/3)}(3\sqrt3+\cos\theta)\Big|_{\pi/6}^{5\pi/6}=\boxed{\frac{\sqrt3}{2(\sqrt3-\pi/3)}}$

$\int_{\pi/6}^{5\pi/6}4\sin^3\theta\,d\theta=4\int_{\pi/6}^{5\pi/6}\sin\theta(1-\cos^2\theta)\,d\theta=\left|\begin{array}{l} u=\cos\theta \\ du=-\sin\theta\,d\theta \\ \theta=\pi/6\to u=\sqrt3/2 \\ \theta=5\pi/6\to u=-\sqrt3/2\end{array}\right|$

$=-4\int_{+\sqrt3/2}^{-\sqrt3/2}(1-u^2)\,du=-4\left(u-\frac{u^3}{3}\right)\Big|_{\sqrt3/2}^{-\sqrt3/2}=-4\left(-\frac{\sqrt3}{2}-\frac{\sqrt3}{2}-\frac13\left(\left(-\frac{\sqrt3}{2}\right)^3-\left(\frac{\sqrt3}{2}\right)^3\right)\right)=$

$-4\left(\frac{-3\sqrt3}{4}\right)=3\sqrt3$

[more problems on back]

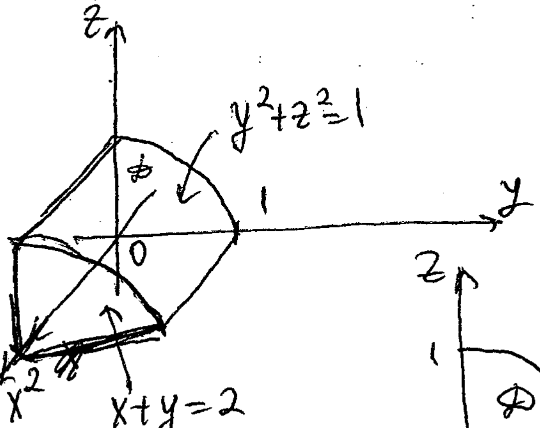
$\boxed{(\bar x,\bar y)=\left(0,\ \frac{\sqrt3}{2(\sqrt3-\pi/3)}\right)}$

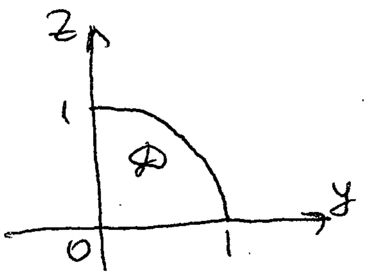
2. *[5 pts.]* Evaluate $\iiint_E z\,dV$, where E is bounded by the planes $y=0$, $z=0$, $x+y=2$ and the cylinder $y^2+z^2=1$ in the first octant.

E: $y=0, z=0$ $\quad$ $x \geq 0$

$x+y=2$ $\quad$ $y \geq 0$

$y^2+z^2=1$ $\quad$ $z \geq 0$





$0 \leq x \leq 2-y$

$0 \leq z \leq \sqrt{1-y^2}$

$0 \leq y \leq 1$

$$\iiint_E z\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{2-y} z\,dx\,dz\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} z(2-y)\,dz\,dy$$

$$= \int_0^1 (2-y)\frac{z^2}{2}\Big|_{z=0}^{z=\sqrt{1-y^2}} dy$$

$$= \frac{1}{2}\int_0^1 (2-y)(1-y^2)\,dy$$

$$= \frac{1}{2}\int_0^1 (2-y-2y^2+y^3)\,dy$$

$$= \frac{1}{2}\left(2y - \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \boxed{\frac{13}{24}}$$

OR:

$0 \leq x \leq 2-y$ $\quad$ $0 \leq r \leq 1$

$y = r\cos\theta$ $\quad$ $0 \leq \theta \leq \pi/2$

$z = r\sin\theta$ $\quad$ $dV = r\,dx\,dr\,d\theta$

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^1 \int_0^{2-r\cos\theta} r^2\sin\theta\,dx\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^2\sin\theta(2-r\cos\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{2r^3}{3}\sin\theta - \frac{r^4}{4}\sin\theta\cos\theta\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\left(\frac{2}{3}\sin\theta - \frac{1}{4}\sin\theta\cos\theta\right)d\theta$$

$$= \frac{2}{3}(-\cos\theta)\Big|_0^{\pi/2} - \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta = \frac{2}{3} - \frac{1}{16}\cos 2\theta\Big|_0^{\pi/2}$$

$$= \frac{2}{3} - \frac{1}{16}(-1-1) = \frac{2}{3} - \frac{1}{8} = \boxed{\frac{13}{24}}$$